1 From patterns to generalizations: sequences and series

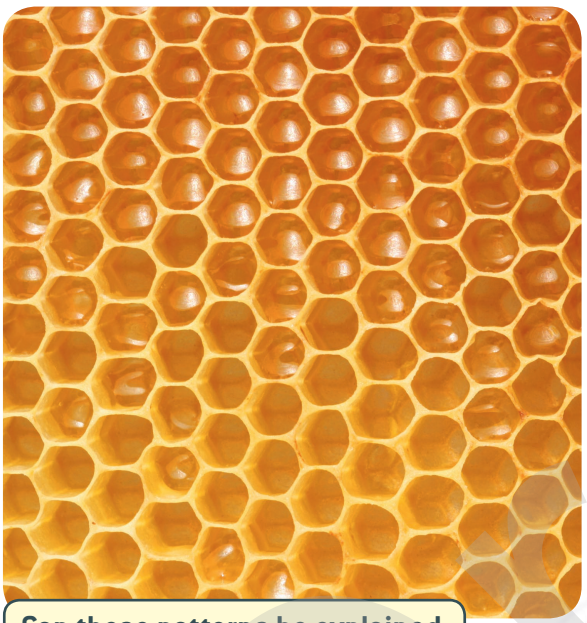
Concepts

- Patterns
- Generalization

Microconcepts

- Arithmetic and geometric sequences
- Arithmetic and geometric series
- Common difference
- Sigma notation
- Common ratio
- Sum of sequences
- Binomial theorem
- Proof
- Sum to infinity

You do not have to look far and wide to find visual patterns—they are everywhere!



Can these patterns be explained mathematically?

Can patterns be useful in real-life situations?

If you take out a loan to buy a car how can you determine the actual amount it will cost?

What information would you require in order to choose the best loan offer? What other scenarios could this be applied to?

The diagrams shown here are the first four iterations of a fractal called the Koch snowflake.

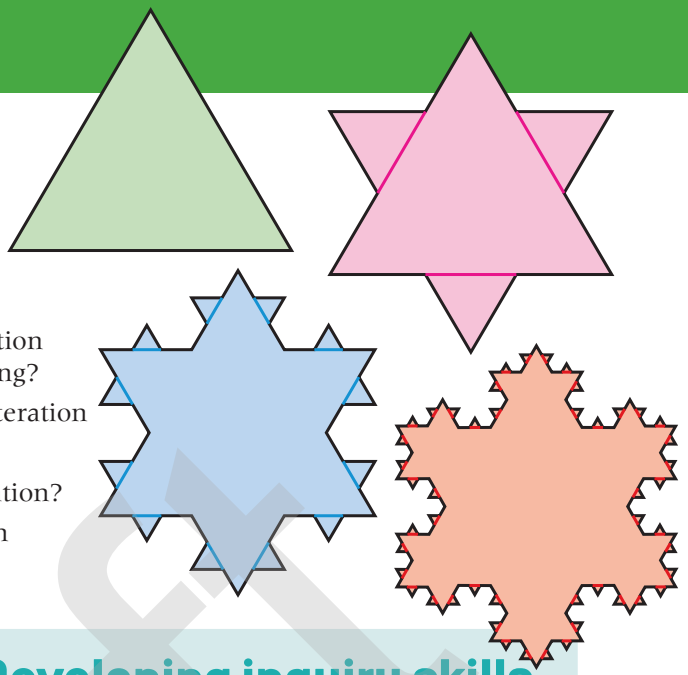
What do you notice about:

- how each pattern is created from the previous one?
- the perimeter as you move from the first iteration through the fourth iteration? How is it changing?
- the area enclosed as you move from the first iteration to the fourth iteration? How is it changing?

What changes would you expect in the fifth iteration?

How would you measure the perimeter at the fifth iteration if the original triangle had sides of 1 m in length?

If this process continues forever, how can an infinite perimeter enclose a finite area?



Developing inquiry skills

Does mathematics always reflect reality? Are fractals such as the Koch snowflake invented or discovered?

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

Click here for help with this skills check

You should know how to:

1 Solve linear equations:

eg $2(x-2)-3(3x+7)=4$

$2x-4-9x-21=4$

$-7x-25=4$

$-7x=29$

$x=\frac{-29}{7}$

2 Perform operations $(+-\times\div)$ with fractions and simplify using order of operations:

eg $\frac{1}{2}+\frac{3}{4}\times\frac{2}{5}$

$=\frac{1}{2}+\frac{6}{20}=\frac{1}{2}+\frac{3}{10}=\frac{5}{10}+\frac{3}{10}=\frac{8}{10}=\frac{4}{5}$

3 Substitute into a formula and simplify using order of operations:

eg $P=-2(x+4)^2+3x,\ x=-1$

$P=-2(-1+4)^2+3(-1)$

$P=-2(3)^2+3(-1)$

$P=-2(9)+3(-1)$

$P=-18-3$

$P=-21$

Skills check

1 Solve each equation:

a $2x+10=-4x-8$

b $3a-2(2a+5)=-12$

c $(x+2)(x-1)=(x+5)(x-2)$

2 Simplify:

a $\frac{3}{4}+\frac{1}{2}-\frac{5}{6}$ **b** $\frac{1}{2}\times\frac{3}{4}+\frac{5}{6}\times\frac{7}{8}$ **c** $\frac{-3}{4}\div\frac{5}{8}$

3 Substitute the given value(s) of the variable(s) and simplify the expression using order of operations:

a $A=2(x-3)^2,\ x=-5$

b $S=2^x+(x-1)(x+3),\ x=2$

c $P=3d-2(n+4)^2,\ d=-3,\ n=2$

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1.1 Number patterns and sigma notation

TOK

Where did numbers come from?

Sequences

Investigation 1

Luis wants to join a gym. He finds one that charges a $100 membership fee to join and $5 every time he uses the facilities. What formula will help Luis to calculate how much he will pay in a given number of visits? If he goes to the gym 12 times in the first month, how much will he pay that month?

Luis's friend Elijah decides to join a different gym. His has no membership fee but charges $15 per use. How many times does each of them need to use their gym before they have paid the same total amount?

1 Complete the table below of Luis's and Elijah's gym fees:

Number of visits to the gym	Total fees paid by Luis	Total fees paid by Elijah
0	$100	$0
1	$105	$15
2	$110	$30
3		
4		
5		
6		
7		
8		

2 What patterns do you see emerging?

3 Which pattern is increasing faster?

4 When will the fees paid by Luis and Elijah be the same?

5 Are there any limitations to these patterns? Will they eventually reach a maximum value?

The fees paid in the investigation above create a sequence of numbers.

A **sequence** (also called a **progression**) is a list of numbers written in a particular order. Each number in a sequence is called a **term**.

Look at the following sequences and see if you can determine the next few terms in the pattern:

i 2, 4, 6, 8, …

ii 14, 11, 8, 5, …

iii 1, 4, 9, 16, …

iv 1, −3, 9, −27, …

A sequence can be either finite or infinite.

A **finite sequence** has a fixed number of terms.

An **infinite sequence** has an infinite number of terms.

For example, 7, 5, 3, 1, –1, –3 is finite because it ends after the sixth term.

For example, 7, 5, 3, –1, … is infinite because the three dots at the end (called an ellipsis) indicate that the sequence is never ending or continues indefinitely.

A sequence can also be described in words. For example, the positive multiples of 3 less than 40 is a finite sequence because the terms are 3, 6, 9, … 39.

A term in a sequence is named using the notation u_n, where n is the position of the term in the sequence. The first term is called u_1, the second term is u_2, etc.

A formula or expression that mathematically describes the pattern of the sequence can be found for the general term, u_n.

Let's return to the opening investigation and determine an expression for the general term for each sequence.

Investigation 1 continued

1. Write down the first few terms for Luis.
2. Describe the pattern.
3. Justify the general expression (u_n) for Luis at the bottom of the table below.
4. Write down the first few terms for Elijah.
5. Describe the pattern.
6. Complete the table below:

Term number	Total fees paid by Luis	Pattern for Luis	Total fees paid by Elijah	Pattern for Elijah
u_1	\$100	$100+(5\times0)$	\$0	15×0
u_2	\$105	$100+(5\times1)$	\$15	
u_3	\$110	$100+(5\times2)$	\$30	
u_4	\$115	$100+(5\times4)$	\$45	
u_5	\$120	$100+(5\times0)$	\$60	
u_n	—	$100+5(n-1)$	—	

7. The general term for Luis can be simplified from $u_n=100+5(n-1)$ to $u_n=5n+95$. Justify this simplification.

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8 What is the simplified form of Elijah's general term?

9 How did the pattern help you to write the general term for Elijah's sequence?

A sequence is called **arithmetic** when the same value is added to each term to get the next term.

For example, $\overset{+5}{\frown}\ \overset{+5}{\frown}\ \overset{+5}{\frown}\ \overset{+5}{\frown}$ 100, 105, 110, 115, 120, …

Another example of an arithmetic sequence is 1, 4, 7, 10, … because 3 is added to each term. The value added could also be negative, such as –5. For example, 20, 15, 10, 5, …

TOK

Do the names that we give things impact how we understand them?

Investigation 2

Let's look at a new type of sequence.

1 Complete this table of values for the next few terms of each sequence:

Sequence 1	Sequence 2	Sequence 3
5	2	36
10	–4	12
20	8	4
40	–16	$\frac{4}{3}$

2 Describe the pattern. For any of these sequences, how do you calculate a term using the previous term?

Look at the pattern for the first sequence as shown in the table below:

Term number	Sequence 1	Pattern	Sequence 2	Pattern	Sequence 3	Pattern
u_1	5	$5\times1=5\times2^0$	2		36	
u_2	10	$5\times2=5\times2^1$	–4		12	
u_3	20	$5\times4=5\times2^2$	8		4	
u_4	40	$5\times8=5\times2^3$	–16		$\frac{4}{3}$	
u_5	80	$5\times16=5\times2^4$	32		$\frac{4}{9}$	
u_n	—	$5\times2^{n-1}$	—		—	

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3 Describe the relationship between the term number and the power of 2 for that term.

4 Justify the expression for the general term for the first pattern.

5 Complete the table for the other two patterns.

6 **Factual** How do you find the general term of **this type** of sequence?

7 **Conceptual** How does looking at the pattern help you find the general term of any sequence?

A sequence is called **geometric** when each term is multiplied by the same value to get the next term.

For example, $5, \overset{\times 2}{\underset{\curvearrowright}{10}}, \overset{\times 2}{\underset{\curvearrowright}{20}}, \overset{\times 2}{\underset{\curvearrowright}{40}}, \ldots$ or $200, \overset{\times \frac{1}{2}}{\underset{\curvearrowright}{100}}, \overset{\times \frac{1}{2}}{\underset{\curvearrowright}{50}}, \overset{\times \frac{1}{2}}{\underset{\curvearrowright}{25}}, \ldots$

Example 1

Find an expression for the general term for each of the following sequences and state whether they are arithmetic, geometric or neither.

a 3, 6, 9, 12, … **b** 3, −12, 48, −192, … **c** 2, 10, 50, 250, … **d** $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots$

a The sequence can be written as $3 \times 1, 3 \times 2, 3 \times 3, \ldots$	This sequence is the positive multiples of 3.
The general term is $u_n = 3n$.	This is an arithmetic sequence because 3 is being added to each term.
This is an arithmetic sequence.	
b The sequence can be written as $3 \times 1, 3 \times -4, 3 \times 16, 3 \times -64$ which can be expressed as $3 \times (-4)^0, 3 \times (-4)^1, 3 \times (-4)^2, 3 \times (-4)^3, \ldots$	This is a geometric sequence because each term is multiplied by −4.
The general term is $u_n = 3(-4)^{n-1}$	
This is a geometric sequence.	
c The sequence can be written as $2 \times 1, 2 \times 5, 2 \times 25, 2 \times 125, \ldots$ or $2 \times 5^0, 2 \times 5^1, 2 \times 5^2, 2 \times 5^3, \ldots$	This is a geometric sequence because each term is multiplied by 5.
The general term is $u_n = 2 \times 5^{n-1}$	
This is a geometric sequence.	

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d The sequence can be written as $\frac{1}{3}, \frac{2}{2+3}, \frac{3}{4+3}, \frac{4}{6+3}, \cdots$ or

$$\frac{1}{3+(2\times 0)}, \frac{2}{3+(2\times 1)}, \frac{3}{3+(2\times 2)}, \frac{4}{3+(2\times 3)}, \ldots$$

The general term is

$$u_n = \frac{n}{3+2(n-1)} = \frac{n}{3+2n-2} = \frac{n}{2n+1}$$

The sequence is neither arithmetic nor geometric.

Consider the numerators and denominators as two separate sequences.

The numerators are the positive integers.

The denominators, 3, 5, 7, 9, … , are the odd positive integers with $u_1 = 3$ and 2 being added to each term.

Each sequence has a first term, a second term, a third term, etc, so n will always be a positive integer.

Exercise 1A

1 Write down the next three terms in each sequence:

a −8, −11, −14, −17, … **b** 9, 16, 25, 36, …

c 6, 12, 18, 24, … **d** 1000, 500, 250, 125, …

e $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ **f** $\frac{1}{3}, \frac{2}{9}, \frac{3}{27}, \frac{4}{81}, \ldots$

2 For each of the following sequences, find an expression for the general term and state whether the sequence is arithmetic, geometric or neither:

a 10, 50, 250, 1250 **b** 41, 35, 29, 23, …

c $\frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, -\frac{1}{81}, \ldots$ **d** 1, 1, 2, 3, 5, 8, …

e $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$ **f** −12, −36, −108, −324, …

3 For each of the following real-life situations, give the general term and state whether the sequence is arithmetic, geometric or neither.

a Ahmad deposits $100 in a savings account every month. After the first month, the balance is $100; after the second month, the balance is $200; and so on.

b Luciana is trying to lose weight. The first month, she loses 6 kg, and she continues to lose half as much each subsequent month.

c The temperature of the water in the swimming pool in your backyard is only 70°F. It is too cold, so you decide to turn up the temperature by 10% every hour until it is warm enough.

Consider the sequence −4, −3, −1, 3, … Notice that it is neither arithmetic or geometric. This is a new type of sequence called recursive.

A **recursive sequence** uses the previous term or terms to find the next term. The general term will include the notation u_{n-1}, which means "the previous term".

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For example, given the general term $u_n = 2u_{n-1} + 5$ and $u_1 = -4$, the second term can be found by replacing u_{n-1} in the general term with u_1:

$$u_2 = 2u_1 + 5 = 2(-4) + 5 = -3$$

The third term can then be found by repeating the process with u_2:

$$u_3 = 2u_2 + 5 = 2(-3) + 5 = -1$$

Here is another example:

Given $u_n = (u_{n-1})^2 - 2$ and $u_1 = 5$, find the next three terms in the sequence.

$$u_2 = (u_1)^2 - 2 = 5^2 - 2 = 23$$
$$u_3 = (u_2)^2 - 2 = 23^2 - 2 = 527$$
$$u_4 = (u_3)^2 - 2 = 527^2 - 2 = 277\,727$$

TOK

Who would you call the founder of algebra?

Example 2

Find the first five terms for each of the following recursive sequences:

a $u_n = \frac{u_{n-1}}{3} + 3$ and $u_1 = 9$ **b** $u_n = 2u_{n-1} - 3$ and $u_1 = 1$

a $u_2 = \frac{u_1}{3} + 3 = \frac{9}{3} + 3 = 3 + 3 = 6$

$u_3 = \frac{u_1}{3} + 3 = \frac{6}{3} + 3 = 2 + 3 = 5$

$u_4 = \frac{u_3}{3} + 3 = \frac{5}{3} + 3 = \frac{5}{3} + \frac{9}{3} = \frac{14}{3}$

$u_5 = \frac{u_4}{3} + 3 = \frac{\frac{14}{3}}{3} + 3 = \frac{14}{9} + \frac{27}{9} = \frac{41}{9}$

$9, 6, 5, \frac{14}{3}, \frac{41}{9}$

To find the second term, replace u_{n-1} in the general term with the value of u_1 given in the question.

To find the third term, now replace u_{n-1} with the answer you found for u_2.

Repeat this process until you have the first five terms.

b $u_2 = 2u_1 - 3 = 2(1) - 3 = -1$
$u_3 = 2u_2 - 3 = 2(-1) - 3 = -5$
$u_4 = 2u_3 - 3 = 2(-5) - 3 = -13$
$u_5 = 2u_4 - 3 = 2(-13) - 3 = -29$

$1, -1, -5, -13, -29$

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Example 3

For each of the recursive sequences below, find a recursive formula for the general term:

a 12, 8, 4, 0, ... **b** −0.32, 3.2, −32, 320, ... **c** 500, 100, 20, 4, ...

a To calculate the next term, add −4.
$u_n = u_{n-1} - 4$, where $u_1 = 12$

b To calculate the next term, multiply by −10.
$u_n = -10u_{n-1}$, where $u_1 = -0.32$

c To calculate the next term, multiply by $\frac{1}{5}$.
$u_n = \frac{1}{5}u_{n-1}$, where $u_1 = 500$

First, consider whether the sequence is arithmetic or geometric. This can be determined by checking whether a value is being added or multiplied to get the consecutive terms.

Since −4 is being added each time, the recursive general term is $u_n = u_{n-1} - 4$, where $u_1 = 12$.

It is important to state the first term so that you know the first value to substitute into the general term.

Exercise 1B

1 Find the first five terms for each of the following recursive sequences.

a $u_n = -4u_{n-1}$, $u_1 = 1$

b $u_n = \frac{-2}{u_{n-1}}, u_1 = 3$

c $u_n = 2(u_{n-1})^2$, $u_1 = -1$

d $u_n = 3u_{n-1} + 5$, $u_1 = m$

2 For each of the recursive sequences below, find a recursive formula for the general term.

a −2, −4, −6, −8, ...

b 1, 4, 64, 256, ...

c 52, 5.2, 0.52, 0.052, ...

d 14, 19, 24, 29, ...

e 2, 3, 6, 18, 108, 1944, ...

f 1, 2, 6, 24, ...

Series

A **series** is created when the terms of a sequence are added together.

A sequence or series can be either finite or infinite.

A **finite series** has a fixed number of terms.

For example, $7 + 5 + 3 + 1 + -1 + -3$ is finite because it ends after the sixth term.

An **infinite series** continues indefinitely.

For example, the series $10 + 8 + 6 + 4 + \ldots$ is infinite because the ellipsis indicates that the series continues indefinitely.

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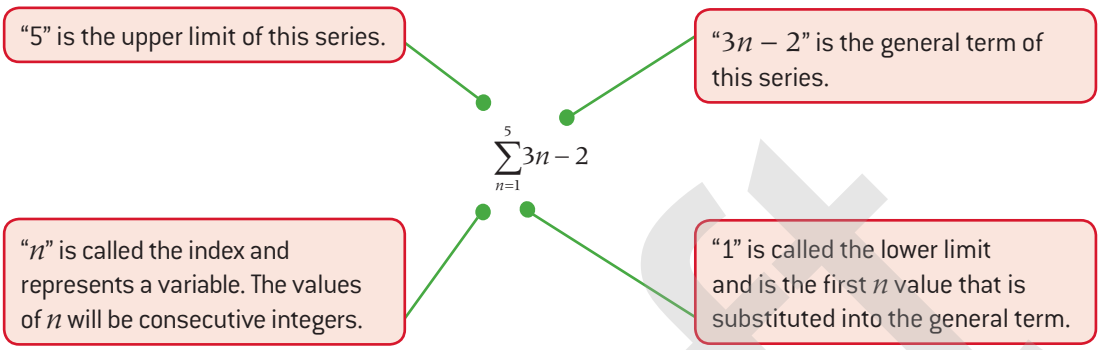
A series can be written in a form called **sigma notation**.

Sigma is the 18th letter of the Greek alphabet, and the capital letter, Σ, is used to represent a sum. Here is an example of a finite series written in sigma notation:



Consecutive n values are substituted, until the "5" or upper limit is reached. It will be the last value substituted.

$$\sum_{n=1}^{5} 3n-2 = (3(1)-2)+(3(2)-2)+(3(3)-2)+(3(4)-2)+(3(5)-2)$$

$$\sum_{n=1}^{5} 3n-2 = 1+4+7+10+13 = 35$$

TOK

Is mathematics a language?

For an infinite series, the upper limit is ∞. An example of an infinite geometric series is $\sum_{n=1}^{\infty} 3\times 2^n$.

Example 4

For each of the following finite series in sigma notation, find the terms and calculate the sum:

a $\sum_{n=1}^{4} (-1)^n n^2$ **b** $\sum_{n=3}^{7} (-2)^n$ **c** $\sum_{n=1}^{3} 2n-n^2$

a $(-1)^1(1)^2 + (-1)^2(2)^2 + (-1)^3(3)^2 + (-1)^4(4)^2$
$= -1 + 4 - 9 + 16$
$= 10$

Substitute $n = 1$ to find the first term, $n = 2$ for the second term and so on.

The last term is found when you substitute the upper limit of $n = 4$.

Remember to add up all the terms once you have found them.

b $(-2)^3 + (-2)^4 + (-2)^5 + (-2)^6 + (-2)^7$
$= (-8) + 16 + (-32) + 64 + (-128)$
$= -88$

Substitute $n = 3$ to find the first term, $n = 4$ for the second term and so.

c $(2(1) - 1^2) + (2(2) - 2^2) + (2(3) - 3^2)$
$= 1 + 0 + (-3)$
$= -2$

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Example 5

Write each of the following series in sigma notation:

a $-3 + 5 + 13 + 21 + 29$ **b** $-3 + 6 - 12 + 24$ **c** $8 + 12 + 16 + 20 + \ldots$

a This finite arithmetic series can be rewritten as:

$-3 + (-3 + 8) + (-3 + 16) + (-3 + 24) + (-3 + 32)$
$= (-3 + 0 \times 8) + (-3 + 1 \times 8) + (-3 + 2 \times 8)$
$+ (-3 + 3 \times 8) + (-3 + 4 \times 8)$

The general term is $-3 + 8n$.

Therefore, the sigma notation is $\sum_{n=0}^{4} -3 + 8n$.

b This finite geometric series can be rewritten as $-3 \times 1 + 3 \times 2 + -3 \times 4 + 3 \times 8$
$= -3 \times 2^0 + 3 \times 2^1 + -3 \times 2^2 + 3 \times 2^3$

The general term must include 3×2^n.

Therefore, the sigma notation is:

$\sum_{n=0}^{3} 3(-1)^{n+1} 2^n$

c $\sum_{n=2}^{\infty} 4n$

or

$\sum_{n=1}^{\infty} 4(n+1)$

Since the first term is negative and the signs of following terms alternate, you need to multiply by $(-1)^{n+1}$.

If the first term was positive, or if you were starting with $n = 1$, you would multiply by $(-1)^n$ to make the signs alternate.

The answers to parts **a** and **b** are just one way to write each sequence using sigma notation. For example, another correct expression for part **a** would be

$\sum_{n=1}^{5} -3 + 8(n-1)$

This infinite arithmetic series is the multiples of 4, starting with the second multiple.

To make the lower limit 1, you can also write the notation like this.

Note that because this is an infinite series, the upper limit is ∞.

Exercise 1C

1 For each of the following series in sigma notation, find the terms and calculate the sum:

a $\sum_{n=1}^{4} (-1)^n (n+1)$ **b** $\sum_{n=2}^{6} 4n - 3$

c $\sum_{n=1}^{3} n(n+1)$ **d** $\sum_{n=3}^{5} \frac{(-1)^{n+1}}{n-2}$

2 Write each of the following series in sigma notation:

a $4 + 16 + 64 + 256 + \ldots$

b $\frac{3}{4} + \frac{4}{5} + \frac{5}{6}$ **c** $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots + \frac{1}{100}$

d $-2 - 2 - 2 - 2 - 2 - 2 - 2 - 2$

e $5 + 10 + 17 + 26 + \ldots$

f $49m^6 + 64m^7 + 81m^8 + 100m^9 + 121m^{10}$

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Developing inquiry skills

Let's return to the chapter opening problem about the Koch snowflake.

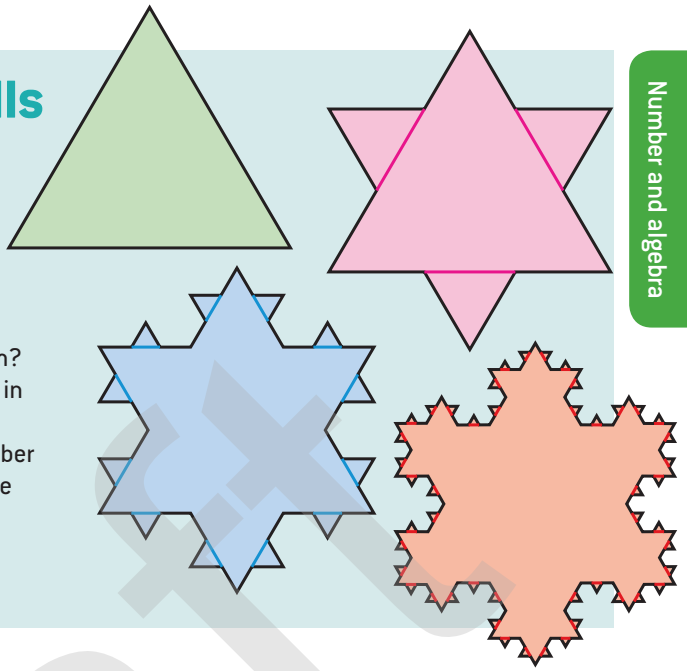
i How many sides does the initial triangle have?

ii How many sides does the second iteration have? What about the third iteration?

iii What kind of sequence do these numbers form?

iv Write the general term for the number of sides in any iteration.

v You can use your general term to find the number of sides for a given iteration by substituting the value for n. Find the number of sides the 12th iteration will have.



1.2 Arithmetic and geometric sequences

Investigation 3

A restaurant is discussing how many people they can fit at their tables. One table seats four people. If they push two tables together, they can seat six people. If three tables are joined, they can seat eight people.

1 Copy and complete the table as shown below:

Number of tables	1	2	3	4	5	6	7	8
Number of seats	4	6	8					

2 For every new table added, how many extra seats are created?

3 Write a general formula to calculate the number of seats for any number of joined tables.

4 You can use your general term to find the number of seats for a given number of tables by substituting the value for n in your general formula. How many seats will there be with twenty tables?

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Let's revisit arithmetic sequences (progressions). From the last section and the investigation above, a sequence is arithmetic when each term after the first is found by adding a fixed non-zero number. This number is called the **common difference (d)**.

The common difference can be found by taking any term and subtracting the previous term: $d = u_n - u_{n-1}$.

When $d > 0$, the sequence is **increasing**.

When $d < 0$, the sequence is **decreasing**.

Let's derive the general formula for any arithmetic sequence:

$u_1, u_2, u_3, \ldots$

$= u_1, u_1 + d, u_1 + d + d, u_1 + d + d + d, \ldots$

$= u_1, u_1 + d, u_1 + 2d, u_1 + 3d, \ldots$

Looking at the pattern between the term number and the coefficient of d, the formula for any term in the sequence is

$$u_n = u_1 + (n - 1)d$$

TOK

Is all knowledge concerned with identification and use of patterns?

Example 6

For each arithmetic progression, write down the value of d and find the term indicated.

a $14, 11, 8, \ldots$ Find u_{12} **b** $2, 7, 12, \ldots$ Find u_9 **c** $-7, -5, -3, \ldots$ Find u_{21}

a $d = 8 - 11$ or $11 - 14 = -3$

$u_n = u1 + (n - 1)d$

$u_{12} = 14 + (12 - 1)(-3)$

$u_{12} = 14 + (11)(-3)$

$u_{12} = 14 - 33$

$u_{12} = -19$

To find d, take any term and subtract the previous term.

Substitute u_1, n and d into the general term formula and simplify.

b $d = 12 - 7$ or $7 - 2 = 5$

$u_n = u_1 + (n - 1)d$

$u_9 = 2 + (9 - 1)(5)$

$u_9 = 2 + (8)(5)$

$u_9 = 2 + 40$

$u_9 = 42$

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c $d = -3 - (-5)$ or $-5 - (-7) = 2$

$u_n = u_1 + (n-1)d$

$u_{21} = -7 + (21-1)(2)$

$u_{21} = -7 + (20)(2)$

$u_{21} = -7 + 40$

$u_{21} = 33$

Exercise 1D

For each sequence below, use the general formula to find the term indicated.

1 5, 13, 21, ... u_9

2 40, 32, 24, ... u_{11}

3 5.05, 5.37, 5.69, ... u_7

4 $\frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \ldots u_6$

5 $x + 2, x + 5, x + 8, \ldots u_9$

6 $3a, 6a, 9a, \ldots u_{12}$

The general formula can be used to find more than just a given term. It can also be used to solve for u_1, n or d.

Example 7

Given an arithmetic sequence in which $u_1 = 14$ and $d = -3$, find the value of n such that $u_n = 2$

$u_n = u_1 + (n-1)d$

$2 = 14 + (n-1)(-3)$

$2 = 14 - 3n + 3$

$-3n = -15$

$n = 5$

Substitute u_1, u_n and d into the general term formula and solve for n.

When solving for n, remember that your answer needs to be a natural number!

Example 8

Find the number of terms in the following finite arithmetic sequence.
$-8, -4, 0, \ldots 36$

$u_n = u_1 + (n-1)d$

$36 = -8 + (n-1)(4)$

$44 = 4n - 4$

$48 = 4n$

$n = 12$

Substitute u_1, u_n and d into the general term formula and solve for n.

Draft

Example 9

Two terms in an arithmetic sequence are $u_6 = 4$ and $u_{11} = 34$. Find u_{15}.

$u_6 + 5d = u_{11}$	The issue here is that you do not have u_1 or d. However, you do know that u_{11} is five terms higher in the sequence than u_6. Therefore, the difference between these terms is $5d$.
$4 + 5d = 34$ $5d = 30$ $d = 6$	Substitute values for the terms and rearrange to find d.
$u_{15} = u_{11} + 4d$ $u_{15} = 34 + 4(6)$ $u_{15} = 58$	Similarly, u_{15} is four terms higher in the sequence than u_{11}.

Exercise 1E

1 Given an arithmetic progression with $u_{21} = 65$ and $d = -2$, find the value of the first term.

2 Given that two terms of an arithmetic sequence are $u_5 = -3.7$ and $u_{15} = -52.3$, find the value of the 19th term.

3 Given an arithmetic sequence in which $u_1 = 11$ and $d = -3$, find the term that has a value of 2.

4 There exists an arithmetic sequence with $t_3 = 4$ and $t_6 = 184$. Find the 14th term.

5 Find the number of terms in the finite arithmetic sequence $6, -1, -8, \ldots, -36$.

6 A movie theatre with 12 rows of seating has 30 seats in the first row. Each row behind it has two additional seats. How many seats are in the last row?

7 The Winter Olympics are held every four years and were held in Vancouver, Canada in 2010. When will the Winter Olympics be held for the first time after 2050?

8 Yinting wants to start adding weight lifting to her exercise routine. Her trainer suggests that she starts with 40 repetitions for the first week and then increases by six repetitions each week after. Find the number of weeks *before* Yinting will be doing 82 repetitions.

Draft

A sequence is **geometric** when each term after the first is found by multiplying by a non-zero number, called the **common ratio (*r*)**.

The common ratio can be found by taking any term and dividing by the previous term: $r = \frac{u_n}{u_{n-1}}$.

When $r < -1$ or $r > 1$, the sequence is **diverging**.

When $-1 < r < -1$, $r \neq 0$, the sequence is **converging**.

The general formula for any geometric sequence can be derived similarly to arithmetic sequences on page 14:

$u_1, u_2, u_3, u_4, u_5, \ldots$

$= u_1, r \times u_1, r \times r \times u_1, r \times r \times r \times u_1, r \times r \times r \times r \times u_1, \ldots$

$= u_1, u_1 r, u_1 r^2, u_1 r^3, u_1 r^4, \ldots$

Looking at the pattern between the term number and the exponent of r, we can see that the formula for any term in the sequence is

$$u_n = u_1 r^{n-1}.$$

EXAM HINT

If $r = 1$, then you will have a constant sequence, not a progression, as all the terms will be the same!

Investigation 4

For each of the given conditions below, generate the first five terms of the geometric sequence using a table:

1 $u_1 = 3, r = 2$

2 $u_1 = -4, r = -2$

3 $u_1 = 100, r = \frac{1}{2}$

4 $u_1 = -7299, r = -\frac{1}{3}$

5 $u_1 = 10\,000, r = \frac{1}{10}$

6 $u_1 = 1, r = 3$

A sequence is called **diverging** if the absolute value of each subsequent term is getting larger, tending away from 0.

A sequence is called **converging** if the absolute value of each subsequent term is getting smaller, approaching 0.

7 **Factual** Sort the sequences into two sets: convergent or divergent.

8 **Conceptual** What do you notice about the value of r in each of the sets?

9 **Conceptual** How can you tell whether a sequence is converging or diverging?

Draft

Example 10

For each geometric progression, write down the value of r and find the term indicated.

a 40, 20, 10, ... u_{12} **b** $\frac{1}{2}, \frac{-1}{4}, \frac{1}{8}, \ldots u_9$ **c** 87, 8.7, 0.87, ... u_6 **d** $7x, 14x^2, 21x^3, \ldots u_8$

a $r = \frac{u_n}{u_{n-1}} = \frac{20}{40} = \frac{1}{2}$

$u_n = u_1 r^{n-1}$

$u_{12} = 40\left(\frac{1}{2}\right)^{12-1}$

$u_{12} = 40\left(\frac{1}{2}\right)^{11}$

$u_{12} = 40\left(\frac{1}{2048}\right)$

$u_{12} = \frac{5}{256}$

b $r = \frac{-\frac{1}{4}}{\frac{1}{2}} = \frac{-1}{2}$

$u_n = u_1 r^{n-1}$

$u_9 = \frac{1}{2}\left(-\frac{1}{2}\right)^{9-1}$

$u_9 = \frac{1}{2}\left(\frac{-1}{2}\right)^{8}$

$u_9 = \frac{1}{512}$

c $r = \frac{u_n}{u_{n-1}} = \frac{8.7}{87} = 0.10$

$u_n = u_1 r^{n-1}$

$u_6 = 87(0.10)^{6-1}$

$u_6 = 87(0.10)^5$

$u_6 = 87(0.00001)$

$u_6 = 0.00087$

d $r = \frac{u_n}{u_{n-1}} = \frac{14x^2}{7x} = 2x$

$u_n = u_1 r^{n-1}$

$u_8 = 7x(2x)^{8-1}$

$u_8 = 7x(2x)^7$

$u_6 = 7x(128x^7)$

$u_6 = 896x^8$

First calculate r by dividing any term by the previous term.

Substitute u_1, n and r into the general term formula and simplify.

Draft

Exercise 1F

1 For the following sequences, determine whether they are geometric and if so, find the term indicated.

a $9, 27, 81, \ldots u_6$ **b** $2, -12, 18, \ldots u_9$

c $6, 4.5, 3.375, \ldots u_7$ **d** $-4, 6, -9, \ldots u_8$

e $500, 100, 20, \ldots u_{13}$ **f** $4, 9, 16, \ldots u_7$

g $3, 3m, 3m^2, \ldots u_{12}$

2 Suppose you find one cent on the first day of September and two cents on the second day, four cents on the third day, and so on. How much money will you find on the last day of September?

3 Write the first five terms of a geometric sequence in which the sixth term is 64.

Example 11

Given $u_1 = 81$ and $u_6 = \frac{1}{729}$ find the values of u_2, u_3, u_4 and u_5 such that the sequence is geometric.

$u_n = u_1 r^{n-1}$

$u_6 = u_1 r^{6-1}$

$\frac{1}{729} = 81r^5$

$r^5 = \frac{1}{59\,049}$

$r = \sqrt[5]{\frac{1}{59\,049}} = \frac{1}{9}$

$u_2 = r \times u_1 = 81\left(\frac{1}{9}\right) = 9$

$u_3 = r \times u_2 = 9\left(\frac{1}{9}\right) = 1$

$u_4 = r \times u_3 = 1\left(\frac{1}{9}\right) = \frac{1}{9}$

$u_5 = r \times u_4 = \left(\frac{1}{9}\right)\left(\frac{1}{9}\right) = \frac{1}{81}$

Therefore, the sequence is

$81, 9, 1, \frac{1}{9}, \frac{1}{81}, \frac{1}{729}$.

This question is asking you to complete the blanks:

$81, ___, ___, ___, ___, \frac{1}{729}$

Note: The term between two existing terms is sometimes called "geometric means".

This makes $u_1 = 81$ and $u_6 = \frac{1}{729}$.

In order to find the missing terms, you first need to find r.

Now you can find the four missing terms by multiplying each term by r, starting with u_2.

Example 12

In a geometric sequence, $u_1 = 5$ and $u_5 = 1\,280$. The last term of the sequence is 20 480. Given that r is positive, find the number of terms in the sequence.

$u_n = u_1 r^{n-1}$

$u_5 = u_1 r^{5-1}$

You first need to find r:

Continued on next page

Draft

$1280 = 5r^4$ $r^4 = 256$ $r = \sqrt[4]{256}$ $r = 4$ $u_n = u_1 r^{n-1}$ $20\,480 = 5 \times 4^{n-1}$ $4\,096 = 4^{n-1}$ $4^6 = 4^{n-1}$ $6 = n - 1$ $n = 7$	Now you can use $u_1 = 5$, $r = 4$ and $u_n = 20\,480$ to find n.

Example 13

The value of a car depreciates at a rate of 19% each year. If you buy a new car today for \$33 560, how much will it be worth after four years?

$r = 100\% - 19\% = 81\% = 0.81$ $u_n = u_1 r^{n-1}$ $u_5 = 33\,560(0.81)^{5-1}$ $u_5 = 33\,560(0.81)^4$ $u_5 = 14\,446.47956\ldots$ The car is worth ≈ \$14 446.48	Since you want to find the value of the car, you need to subtract the depreciation from 100% to find r. Since u_1 is when time = 0, after four years would be the fifth term. Note: Be sure to use approximation signs for any rounded values. Unless otherwise stated, monetary values should be rounded to two decimal places.

Exercise 1G

1 If a geometric sequence has $u_5 = 40$ and $u_{10} = 303.75$, find the value of u_{15}.

2 For a geometric sequence that has $r = -\frac{4}{5}$ and $u_6 = -1\,280$, find the 20th term.

3 If 16, $x + 2$, 1 are the first three terms of a geometric sequence, find all possible values of x.

4 Find the number of terms in the geometric sequence

6, 12, 24, …, 1 536

5 Find the possible values of the common ratio of a geometric sequence whose first term is 2 and whose fifth term is 32.

6 In 2017, the number of students enrolled in a high school was 232. It is estimated that the student population will increase by 3% every year. Estimate the number of students that will be enrolled in 2027.

7 An old legend states that a peasant won a reward from a king. The peasant asked to be paid in rice; one grain on the first square of a chessboard, two grains on the second, four on the third square, and so on.

Draft

a How many grains of rice would be on the 30th square?

b Which square would contain exactly 512 grains of rice?

8 Given $u_1 = 8$ and $u_5 = 128$, find the values of u_2, u_3 and u_4 such that the sequence is geometric.

Investigation 5

Consider each of the sequences below:

a 20, 16, 12, ...

b 20, 10, 5, ...

c −4, −12, −36, ...

d −16, −18, −20, ...

e 1, 2, 6, ...

f $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

g 2, 4, 6, ...

h 3, 6, 7, ...

1 Decide whether each sequence is geometric, arithmetic or neither.

2 Find the general term for each sequence, if possible.

3 **Factual** What is a common difference? What is a common ratio?

4 **Conceptual** How can you determine whether a sequence is geometric, arithmetic or neither?

5 **Conceptual** How do the common difference or common ratio help you to describe and generalize a sequence?

Exercise 1H

For each question below, first decide whether the situation is arithmetic or geometric, then solve accordingly.

1 A frog fell into a 1 m well and wanted to go back up to the top of the well. Every day it moved up half the distance to the top. After 10 days, how much did the frog have left to climb?

2 Your grandparents deposit $2000 into a bank account to start a college fund for you. They will continue to deposit a fixed amount each month if you deposit $5 a month as well. In 36 months, you would like to have $6500 in the account. How much will they have to contribute each month?

3 The Chinese zodiac associates years with animals, based on a 12-year cycle. Samu was born in 1962, the Year of the Tiger. He lives in Finland, which celebrated its centennial in 2017. Did Finland gain its independence in the Year of the Tiger?

4 A scientist puts six bacteria, which multiply at a constant rate, in a Petri dish. She records the number of bacteria each minute thereafter. If she counts 324 bacteria 20 minutes later, at what rate are the bacteria reproducing?

Draft

1.3 Arithmetic and geometric series

Arithmetic series

Investigation 6

Let's make a triangle out of coins. Place one coin on the table.

Underneath it, create a row of two coins.

Place three in the next row.

Continue adding rows, each with one more coin than the row before.

1 Is the sequence between the row number and number of coins in that row arithmetic or geometric?

2 Write the general formula for the number of coins in any row.

3 Complete the table showing the **total number** of coins it will take to make a triangle with the given number of rows:

Number of rows	Total number of coins
1	1
2	3
3	
4	
5	

4 Hypothesize how many coins are needed to create a triangle with 10 rows. How many coins are needed for 20 rows? How many coins are needed for n rows?

The sum of the terms of an arithmetic sequence is called an **arithmetic series**.

An example of an arithmetic series is $3 + 6 + 9 + 12 + \ldots 30$.

Carl Gauss, a German mathematician, was the first to determine the formula to calculate the sum of an arithmetic sequence. When he tried to add up the natural numbers from 1 to 100, he noticed he could match them into pairs with equal sum:

TOK

How is intuition used in mathematics?

$$1 + 2 + 3 + \cdots 98 + 99 + 100$$

$1 + 100 = 101$ $\quad$ $2 + 99 = 101$ $\quad$ $3 + 98 = 101$

Draft

Since there would be 50 pairs whose sum is 101, the total sum must be $50 \times 101 = 5050$.

Consider the same process with the general arithmetic series:

$$u_1 + u_2 + u_3 + \ldots u_n$$

We can match the pairs into equal sums:

$$u_1 + u_n \qquad u_2 + u_{(n-1)} \qquad u_3 + u_{(n-2)}$$

Since we started with n terms, there would be $\frac{n}{2}$ sums.

The sum, S_n, would be expressed as $S_n = \frac{n}{2}(u_1 + u_n)$.

Since we already have the formula $u_n = u_1 + (n-1)d$, we can substitute it into the sum formula:

$$S_n = \frac{n}{2}(u_1 + u_n) = \frac{n}{2}(u_1 + u_1 + (n-1)d) = \frac{n}{2}(2u_1 + (n-1)d)$$

This gives us a second formula for the sum:

$$S_n = \frac{n}{2}[2u_1 + (n-1)d]$$

Geometric series

For geometric series, $u_1 + u_1 r + u_1 r^2 + \cdots u_1 r^{n-2} + u_1 r^{n-1}$, the formula for the sum is:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, \; r \neq 1$$

Example 14

For each infinite series below, decide whether it is arithmetic or geometric, then find the sum to the given term:

a $1 + 5 + 9 + \ldots u_8$

b $6 + 12 + 18 + \ldots u_7$

c $-2 - 4 - 8 - \ldots u_9$

d $100 + 50 + 25 + \ldots u_6$

e $132 + 124 + 116 + \ldots u_7$

f $\frac{1}{8} - \frac{1}{4} + \frac{1}{2} - \ldots u_{12}$

Continued on next page

Draft

a Arithmetic

Since the difference between each term is +4, this series is arithmetic.

$d = u_n - u_{n-1}$

$d = 5 - 1 = 4$

Find the value of d.

$S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$

$S_8 = \frac{8}{2}\left[2(1) + (8-1)(4)\right]$

$S_8 = 4[2 + (7)(4)]$

$S_8 = 4[2 + 28]$

$S_8 = 4(30)$

$S_8 = 120$

Substitute n, u_1 and d into the sum formula and simplify.

b Arithmetic

Since the difference between each term is +6, this series is arithmetic.

$d = u_n - u_{n-1}$

$d = 12 - 6 = 6$

Find the value of d.

$S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$

$S_7 = \frac{7}{2}\left[2(6) + (7-1)(6)\right]$

$S_7 = 3.5[12 + (6)(6)]$

$S_7 = 3.5[12 + 36]$

$S_7 = 3.5(48)$

$S_7 = 168$

Substitute n, u_1 and d into the sum formula and simplify.

c Geometric

Since each term is found by multiplying the previous term by 2, this series is geometric.

$r = \frac{u_n}{u_{n-1}} = \frac{-4}{-2} = 2$

Find the value of r.

$S_n = u_1\left(\frac{1 - r^n}{1 - r}\right)$

$S_9 = -2\left(\frac{1 - 2^9}{1 - 2}\right)$

$S_9 = -2\left(\frac{1 - 512}{-1}\right)$

$S_9 = -2(511)$

$S_9 = -1022$

Substitute n, u_1 and r into the sum formula and simplify.

Draft

d Geometric

Since each term is found by multiplying the previous term by $\frac{1}{2}$, this series is geometric.

$r = \frac{50}{100} = \frac{1}{2}$

Find the value of r.

$S_n = u_1\left(\frac{1-r^n}{1-r}\right)$

Substitute n, u_1 and r into the sum formula and simplify.

$S_6 = 100\left(\frac{1-\left(\frac{1}{2}\right)^6}{1-\frac{1}{2}}\right)$

$S_6 = 100\left(\frac{1-\frac{1}{64}}{\frac{1}{2}}\right)$

$S_6 = 100\left(\frac{63}{64}\times\frac{2}{1}\right)$

$S_6 = 100\left(\frac{63}{32}\right)$

$S_6 = \frac{6300}{32} = \frac{1575}{8}$

e Arithmetic

Since the difference between each term is −8, this series is arithmetic.

$d = u_n - u_{n-1}$

Find the value of d.

$d = 124 - 132 = -8$

$S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$

Substitute n, u_1 and d into the sum formula and simplify.

$S_7 = \frac{7}{2}\left[2(132) + (7-1)(-8)\right]$

$S_7 = 3.5[264 + (6)(-8)]$

$S_7 = 3.5(264 - 48)$

$S_7 = 3.5(216)$

$S_7 = 756$

Continued on next page

Draft

f Geometric	Since each term is found by multiplying the previous term by –2, this series is geometric.
$r = \frac{-\frac{1}{4}}{\frac{1}{8}} = -2$	Find the value of r.
$S_n = u_1\left(\frac{1-r^n}{1-r}\right)$	Substitute n, u_1 and r into the sum formula and simplify.
$S_{12} = \frac{1}{8}\left(\frac{1-(-2)^{12}}{1-(-2)}\right)$	
$S_{12} = \frac{1}{8}\left(\frac{1-4096}{3}\right)$	
$S_{12} = \frac{1}{8}(-1365)$	
$S_{12} = \frac{-1365}{8}$	

You can also use these formulas to find the sums of finite series.

Example 15

For each finite series below, decide whether it is arithmetic or geometric, then find the sum:

a $5 + 7 + 9 + \cdots 39$ **b** $\sum_{n=1}^{15} -2(3)^n$ **c** $\frac{5}{4} + \frac{5}{8} + \frac{5}{16} + \cdots \frac{5}{256}$

a Arithmetic	Since the difference between each term is 2, this series is arithmetic.
$d = u_n - u_{n-1}$	
$d = 7 - 5 = 2$	
$u_n = u_1 + (n-1)d$	Since you do not have n, you need to find this first using the general term formula.
$39 = 5 + (n-1)(2)$	
$34 = 2(n-1)$	
$17 = n - 1$	
$n = 18$	
$S_{18} = \frac{18}{2}(5 + 39)$	Substitute n, u_1 and u_n into the sum formula and simplify.
$S_{18} = 9(44)$	
$S_{18} = 396$	

Draft

b Geometric

$$r = \frac{-2(3)^2}{-2(3)} = 3$$

$$S_{15} = -6\left(\frac{1-3^{15}}{1-3}\right)$$

$$S_{15} = -6\left(\frac{1-14\,348\,907}{-2}\right)$$

$$S_{15} = 3(-14\,348\,906)$$

$$S_{15} = -43\,046\,718$$

Since each term is found by multiplying the previous term by 3, this series is geometric.

Find the value of r.

Substitute n, u_1 and r into the sum formula and simplify.

c Geometric

$$r = \frac{u_n}{u_{n-1}} = \frac{\frac{5}{8}}{\frac{5}{4}} = \frac{1}{2}$$

$$u_n = u_1 r^{n-1}$$

$$\frac{5}{256} = \frac{5}{4}\left(\frac{1}{2}\right)^{n-1}$$

$$\frac{1}{64} = \left(\frac{1}{2}\right)^{n-1}$$

$$\left(\frac{1}{2}\right)^6 = \left(\frac{1}{2}\right)^{n-1}$$

$$n - 1 = 6$$

$$n = 7$$

$$S_7 = \frac{5}{4}\left(\frac{1-\left(\frac{1}{2}\right)^7}{1-\frac{1}{2}}\right)$$

$$S_7 = \frac{5}{4}\left(\frac{1-\frac{1}{128}}{\frac{1}{2}}\right)$$

$$S_7 = \frac{5}{4}\left(\frac{127}{128} \times \frac{2}{1}\right)$$

$$S_7 = \frac{5}{4}\left(\frac{127}{64}\right)$$

$$S_7 = \frac{635}{256}$$

Since each term is found by multiplying the previous term by $\frac{1}{2}$, this series is geometric.

Find the value of r.

Since you do not have n, you need to find this first using the general term formula.

Substitute n, u_1 and r into the sum formula and simplify.

Draft

Exercise 1I

1 For each series below, decide whether it is arithmetic or geometric, then find the sum.

a $\frac{1}{5}+\frac{8}{15}+\frac{13}{15}+\cdots\ S_7$

b $\sum_{n=1}^{8}\frac{1}{2}(-3)^n$

c $0.1 + 0.05 + 0.025 + \ldots\ S_8$

d $6 + 12 + 18 + \ldots\ 288$

e All multiples of 4 between 1 and 999

f $\sum_{n=1}^{6}(-1)^{n-1}(2)^n$

2 A theatre has 30 rows of seats. There are 22 seats in the first row, 26 in the second row, 30 in the third row, etc. How many people will the theatre hold?

3 Every human has two biological parents, four biological grandparents, eight biological great-grandparents and so on. How many biological ancestors are in your family for the last six generations if you are the first generation?

4 Each hour, a grandfather clock chimes the number of times that corresponds to the time of day. For example, at 5 am, it will chime five times. How many times will the clock chime in 24 hours?

5 Consider the following sequence of figures made up of line segments.

Figure 1 Figure 2 Figure 3

Find the total number of line segments in the first 48 figures.

Let's return to the coin triangle investigation from the beginning of this section.

Can you now find the total number of coins needed to form a triangle with 10 rows? What about for 20 rows?

Investigation 7

Let's return to infinite geometric series.

Recall from section 1.2 that a geometric sequence is diverging if $|r| > 1$ and is converging if $|r| < 1$. This holds true for series as well.

Draft

Part 1:

1 Copy and complete the chart below for a geometric series with $u_1 = 1$ and $r = 0.5$:

Term number	Term	Sum
1	1	1
2	0.5	1.5
3	0.25	1.75
4		
5		
6		
7		
8		
9		
10		
—	—	—

2 As the term number increases, what do you notice about the value of the term?

3 Looking at the second column, is the sequence converging or diverging?

4 How does this affect the sum?

5 What value is the sum approaching?

6 Using your GDC, plot the first sequence with the term number on the x-axis and the value of the term on the y-axis.

7 How does a graph show that the sum of a sequence is converging? What value is the sum of the first sequence converging to?

Part 2:

1 Repeat steps above for the following sequences:

a $u_1 = 10,\ r = \frac{1}{10}$ **b** $u_1 = -3,\ r = 2$

c $u_1 = 248,\ r = -\frac{1}{2}$ **d** $u_1 = 1,\ r = 4$

2 **Conceptual** When does an infinite geometric sequence have a finite sum?

3 Find the first four terms of the arithmetic sequence below:

a $u_1 = 10,\ d = 5$ **b** $u_1 = 10,\ d = -2$

c $u_1 = 10,\ d = -0.5$ **d** $u_1 = 10,\ d = 0.4$

4 **Conceptual** Which of the arithmetic sequences are decreasing? Which are increasing?

5 **Conceptual** Does an infinite arithmetic sequence converge to a finite sum?

6 **Conceptual** When does an infinite sequence have a finite sum?

Draft

Let's look at the sum of converging geometric series.

Consider the formula for the sum of a finite geometric series:

$$S_n = \frac{u_1(1-r^n)}{1-r}$$

Distributing the numerator:

$$S_n = \frac{u_1 - u_1 r^n}{1-r}$$

Splitting into two fractions:

$$S_n = \frac{u_1}{1-r} - \frac{u_1 r^n}{1-r}$$

As you saw in the previous investigation, with a converging geometric sequence, as the term number increases, the value of the term becomes increasingly smaller, which has almost no impact on the sum. Therefore, as the number of terms approaches infinity, the limit of $\frac{u_1 r^n}{1-r}$ will approach zero.

The sum of a converging infinite geometric series is $S_\infty = \frac{u_1}{1-r}, |r| < 1$

Example 16

Decide whether the infinite geometric series below are converging, and if so, find the sum:

a $20 + 10 + 5 + \cdots$ **b** $\frac{1}{3} + \frac{2}{3} + \frac{4}{3} + \cdots$ **c** $\frac{1}{4} - \frac{1}{8} + \frac{1}{16} + \cdots$ **d** $\sum_{k=1}^{\infty} -2\left(\frac{-1}{3}\right)^n$

a $r = \frac{u_n}{u_{n-1}} = \frac{10}{20} = \frac{1}{2}$

Converging

$S_\infty = \frac{u_1}{1-r}$

$S_\infty = \frac{20}{1-\frac{1}{2}}$

$S_\infty = \frac{20}{\frac{1}{2}}$

$S_\infty = 20 \times \frac{2}{1}$

$S_\infty = 40$

This series is converging since $|r| < 1$.

b $r = \frac{u_n}{u_{n-1}} = \frac{\frac{2}{3}}{\frac{1}{3}} = \frac{2}{3} \times \frac{3}{1} = 2$

Not converging

This series is not converging since $r > 1$.

c $r = \frac{u_n}{u_{n-1}} = \frac{-\frac{1}{8}}{\frac{1}{4}} = -\frac{1}{8} \times \frac{4}{1} = -\frac{1}{2}$

This series is converging since $|r| < 1$.

Draft

Converging

$S_\infty = \frac{u_1}{1-r}$

$S_\infty = \frac{\frac{1}{4}}{1--\frac{1}{2}}$

$S_\infty = \frac{\frac{1}{4}}{1+\frac{1}{2}}$

$S_\infty = \frac{\frac{1}{4}}{\frac{3}{2}}$

$S_\infty = \frac{1}{4} \times \frac{2}{3}$

$S_\infty = \frac{1}{6}$

d $u_1 = -2\left(\frac{-1}{3}\right)^1 = \frac{2}{3}$

$u_2 = -2\left(\frac{-1}{3}\right)^2 = \frac{-2}{9}$

$r = \frac{u_n}{u_{n-1}} = \frac{-\frac{2}{9}}{\frac{2}{3}} = -\frac{2}{9} \times \frac{3}{2} = -\frac{1}{3}$

Converging

This series is converging since $|r| < 1$.

$S_\infty = \frac{\frac{2}{3}}{1-\left(-\frac{1}{3}\right)}$

$S_\infty = \frac{\frac{2}{3}}{1+\frac{1}{3}}$

$S_\infty = \frac{\frac{2}{3}}{\frac{4}{3}}$

$S_\infty = \frac{2}{3} \times \frac{3}{4}$

$S_\infty = \frac{1}{2}$

Draft

Example 17

A bouncy ball is dropped straight down from a height of 1.5 metres. With each bounce, it rebounds 75% of its height. What is the total distance it travels until it comes to rest?

$u_1 = 1.5$ $r = 0.75$ $S_\infty = ?$ $S_\infty = \frac{u_1}{1-r} = \frac{1.5}{1-0.75}$ $S_\infty = \frac{1.5}{0.25}$ $S_\infty = 6$	
Distance travelled = $2 \times 6 = 12$	However, this only represents the distance travelled on the down bounces. Because the ball also travels up, you must double the sum.
Distance travelled = 12 − 1.5 = 10.5 metres	But since the ball was dropped from above, you must subtract the original 1.5 metres.

Example 18

Avi does chores around his house to earn money to buy his family presents for Hanukkah. On the first day of December, he earns \$0.50. On the second, he earns \$1.75 and on the third, he earns \$3. If this pattern continues for the first two weeks of December, how much money will he have in total on the first night of Hanukkah on 14 December?

The sequence is 0.50, 1.75, 3, … $d = u_n - u_{n-1}$ $d = 1.75 - 0.50 = 1.25$ $S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$ $S_{14} = \frac{14}{2}\left[2(0.50) + (14-1)(1.25)\right]$ $S_{14} = 7[1 + (13)(1.25)]$ $S_{14} = \$120.75$	

TOK

Is it possible to know things about which we can have no experience, such as infinity?

Draft

Exercise 1J

1 Decide whether the infinite geometric series below are converging, and if so, find the sum:

a $0.25 + 0.375 + 0.5625 + \ldots$

b $-\frac{3}{8} + \frac{9}{32} - \frac{27}{128} + \ldots$

c $\sum_{n=1}^{\infty} -5\left(\frac{1}{2}\right)^n$

d $\sum_{n=1}^{\infty} (-1)^{n+1}(2)^n$

e $(27x - 27) + (9x - 9) + (3x - 3) + \ldots$

f $\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}} + \cdots$

g $\frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} + \frac{1}{4\sqrt{2}} + \cdots$

2 Give an example of an infinite geometric series with a finite sum.

3 Give an example of an infinite geometric series where it is impossible to find the sum.

4 A ball is dropped from a height of 12 ft. Each time it hits the ground, it rebounds to $\frac{3}{5}$ of its previous height. Find how far the ball will travel before coming to a stop.

5 An oil well in Texas, USA produces 426 barrels in its first day of production. Its production decreases by 0.1% each day. If there are 42 gallons in one barrel of oil, find how many gallons this well will produce before it runs dry.

Example 19

For an arithmetic series, $S_n = -126$, $u_1 = -1$ and $u_n = -20$. Find the value of n.

$S_n = \frac{n}{2}[u_1 + u_n]$ $-126 = \frac{n}{2}[-1 - 20]$ $-252 = -21n$ $n = 12$	Remember that there are two formulas for the sum of an arithmetic series. Since you are given u_n, use the formula $S_n = \frac{n}{2}(u_1 + u_n)$

Example 20

The second term of an arithmetic sequence is 7. The sum of the first four terms is 12. Find the first term and the common difference.

$u_2 = 7$ $S_4 = 12$ $7 = u_1 + (2 - 1)d$ $7 = u_1 + d$ $u_1 = 7 - d$	

Continued on next page

Draft

$12 = \frac{4}{2}\left[2u_1 + (4-1)d\right]$ $12 = 2[2u_1 + 3d]$ $12 = 2[2(7-d) + 3d]$ $12 = 2[14 - 2d + 3d]$ $12 = 2[14 + d]$ $6 = d + 14$ $d = -8$ $u_1 = 7 - d = 7 - (-8) = 15$	

Example 21

For the series $3 + 9 + 27 + \ldots$, what is the minimum number of terms needed for the sum to exceed 1000?

$\frac{3(3^n - 1)}{3-1} > 1000$ Using the GDC: $n = 5.92\ldots$ The minimum number of terms is 6. or $\frac{3(3^n - 1)}{3-1} > 1000$ Using a table on the GDC: when $n = 5$, $S_n = 363$ when $n = 6$, $S_n = 1092$ The minimum number of terms is 6.	When using a graph or solver on the GDC, be sure to give the decimal answer first, then give the final answer. When using a table on the GDC, be sure to give both crossover values, then give the final answer.

Exercise 1K

1 In an arithmetic sequence, the first term is −8 and the sum of the first 20 terms is 790.

a Find the common difference.

b **i** Find u_{28}.

ii Hence, find S_{28}.

c Find how many terms it takes for the sum to exceed 2000.

Draft

2 In an arithmetic series, $S_{40} = 1900$ and $u_{40} = 106$. Find the value of the first term and the common difference.

3 The sum of an infinite geometric series is 20, and the common ratio is 0.2. Find the first term of this series.

4 The sum of an infinite geometric series is three times the first term. Find the common ratio of this series.

5 Create two different infinite geometric series that each have a sum of 8.

6 In a geometric sequence, the fourth term is 8 times the first term. The sum of the first 10 terms is 2557.5. Find the 10th term of this sequence.

7 A large company created a phone tree to contact all employees in case of an emergency. Each of the five vice presidents calls five employees, who in turn each call five other employees, and so on. How many rounds of phone calls are needed to reach all 2375 employees?

8 A geometric sequence has all positive terms. The sum of the first two terms is 15 and the sum to infinity is 27.

a Find the value of the common ratio.

b Hence, find the first term.

9 The first three terms of an infinite geometric sequence are $m - 1$, 6, $m + 8$.

a Write down two expressions for r.

b **i** Find two possible values of m.

ii Hence, find two possible values of r.

c **i** Only one of these r values forms a geometric sequence where an infinite sum can be found. Justify your choice for r.

ii Hence, calculate the sum to infinity.

TOK

How do mathematicians reconcile the fact that some conclusions conflict with intuition?

Developing inquiry skills

Returning to the chapter opening investigation about the Koch snowflake, the enclosed area can be found using the sum of an infinite series.

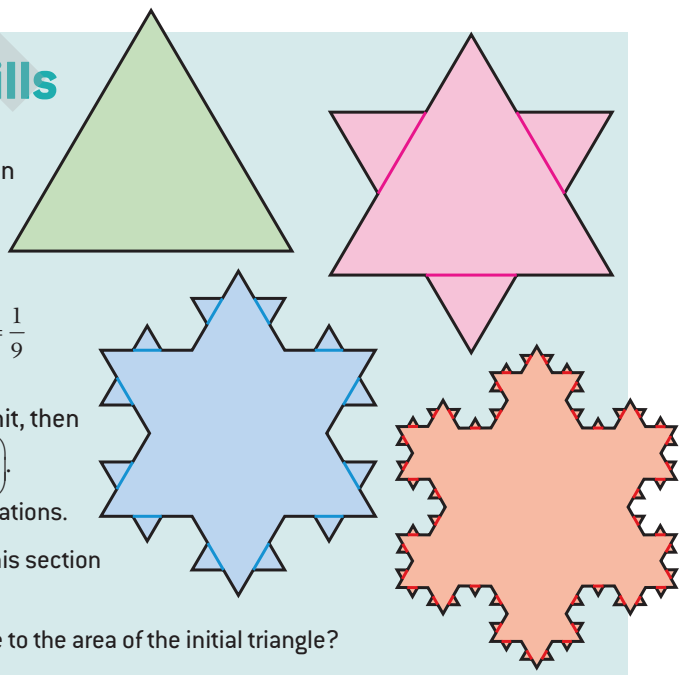
In the second iteration, since the sides of the new triangles are $\frac{1}{3}$ the length of the sides of the original triangle, their areas must be $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$ of its area.

If the area of the original triangle is 1 square unit, then the total area of the three new triangles is $3\left(\frac{1}{9}\right)$.

i Find the total area for the third and fourth iterations.

ii How can you use what you have learned in this section to find the total area of the Koch snowflake?

iii How does the area of a Koch snowflake relate to the area of the initial triangle?



Draft

1.4 Applications of arithmetic and geometric patterns

As you have seen in previous sections, arithmetic and geometric sequences and series can be applied to many real-life situations. In this section, you will explore two of these applications: interest and population growth.

Interest is the charge for borrowing money. Interest can also be used to describe money earned on an investment account. You will focus on two types of interest, simple and compound.

Simple interest is interest paid on the initial amount borrowed, saved or invested (called the **principal**) only and not on past interest.

Compound interest is paid on the principal and the accumulated interest. Interest paid on interest!

Investigation 8

Part 1:
An amount of $1000 is invested into an account that pays 5% per annum simple interest on a monthly basis.

1 Calculate the amount of interest paid per month.

2 Find the value of this investment after the first three months.

3 What do you notice about this sequence? What sort of sequence is it?

4 Write the general formula that can be used to find the amount of the investment after n months.

5 Use your formula to calculate the value of investment after two years.

Part 2:
An amount of $1000 is invested into an account that pays 5% per annum compound interest. Interest is compounded monthly.

1 Calculate the amount of interest paid after the first month.

2 Find the value of the investment after the first month.

3 Continue this process for the second and third months.

4 What do you notice about this sequence? What sort of sequence is it?

5 Write the general formula that can be used to find the amount of the investment after n months.

6 Use your formula to calculate the value of investment after two years.

7 Compare your answers from the final question of each part. Which investment is worth more; the simple interest or the compound interest?

Draft

8 **Conceptual** Which type of series models simple interest? Which type models compound interest?

9 **Conceptual** How are outcomes and growth patterns related?

From the previous investigation, you should have concluded that simple interest can be modelled by an arithmetic series, and compound interest can be modelled by a geometric series.

For simple interest, assuming A represents the accumulated amount, P is the principal, r is the annual rate and n is the time in years:

Initial amount:
$A = P$

After the first year:
$A = P + Pr = P(1 + r)$

After the second year:
$A = P(1 + r) + Pr = P(1 + r + r) = P(1 + 2r)$

After the third year:
$A = P(1 + 2r) + Pr = P(1 + 2r + r) = P(1 + 3r)$

After the nth year:
$\boldsymbol{A = P(1 + nr)}$

For compound interest, making the same assumptions as above:

Initial amount:
$A = P$

After the first year:
$A = P + Pr = P(1 + r)$

After the second year:
$A = P(1 + r) + P(1 + r)r = P(1 + r)(1 + r) = P(1 + r)^2$

After the third year:
$A = P(1 + r)^2 + P(1 + r)^2 r = P(1 + r)^2(1 + r) = P(1 + r)^3$

After the nth year:
$\boldsymbol{A = P(1 + r)^n}$

Interest can be paid over any time period—for example, yearly, monthly, quarterly, and so on.

To calculate compound interest over a non-year period, we use this formula:

$A = P\left(1 + \frac{r}{n}\right)^{nt}$, where A is the final amount (principal + interest), P is the principal, r is the annual interest rate expressed as a decimal, n is the number of compoundings in a year, and t is the total number of years.

Draft

Example 22

Sebastian took out a loan for a new car that cost $35 000. The bank offered him 2.5% per annum simple interest for five years. Calculate the total value Sebastian has to repay the bank.

$A = P(1 + nr)$

$A = 35\,000(1 + 5(0.025))$

$A = 35\,000(1.125)$

$A = \$39\,375$

Example 23

Habib put $5000 into a savings account that pays 4% interest per annum, compounded monthly. How much will be in the account after four years if Habib does not deposit or withdraw any money?

Since the interest is paid monthly,

$r = \frac{0.04}{12}$ and

$nt = 4 \times 12 = 48$

$A = 5000\left(1 + \frac{0.04}{12}\right)^{48}$

$A \approx \$5865.99$

Example 24

Leslie got a student loan of $12 090 CAD to pay her tuition at the University of Toronto this year. If Leslie has to repay $16 000 CAD in two years, what is the annual interest rate if the interest on the loan is compounded monthly?

$16\,000 = 12\,090\left(1 + \frac{r}{12}\right)^{24}$

$1.3234\ldots = \left(1 + \frac{r}{12}\right)^{24}$

$1 + \frac{r}{12} = \sqrt[24]{1.3234\ldots}$

$1 + \frac{r}{12} = 1.0117438\ldots$

$\frac{r}{12} = 0.0117438\ldots$

$r = 0.14092\ldots$

$r \approx 14.1\%$

Draft

Example 25

Oliver's savings account now has a balance of $900. If the interest rate was 1.5% per annum, compounded quarterly, how much did Oliver originally deposit six years ago if he has not withdrawn or deposited any money since?

If the interest is compounded quarterly,

$r = \frac{0.015}{4}$

and $nt = 6 \times 4 = 24$

$900 = P\left(1 + \frac{0.015}{4}\right)^{24}$

$900 = P(1.00375)^{24}$

$900 = P(1.09399\ldots)$

$P = \$822.6766\ldots$

$P \approx \$822.68$

Exercise 1L

1 For each final amount given below, calculate the amount of interest paid/earned:

- **a** 1500 USD, simple interest of 6%, paid annually for ten years.
- **b** 32 000 GBP, simple interest of 1.25%, paid for eight years.
- **c** 14 168 000 Yen, compound interest of 2%, compounded monthly for three years.
- **d** 300 000 Mexican Peso, compound interest of 4%, compounded daily for two years.
- **e** 250 000 Swiss Francs, compound interest of 2.25%, compounded monthly for 25 years.

2 Fernando wants to purchase a new laptop for 2 323 000 Columbian Pesos. The electronics store has a financing offer. Fernando pays the electronics store in weekly installments over two years.

- **i** What is the simple interest rate the electronics store is charging if he ends up paying it 2 480 000 Pesos in total?
- **ii** How much is his weekly payment to the electronics store? Round your answer to the nearest Peso.

3 A teacher makes a one-time investment of $90 000 in a retirement account for five years. The annual interest rate is 2.25%, compounded monthly. What will the final balance of the account be?

4 After five years of making quarterly payments in her car loan, Riley has paid off her loan. She has paid a total of $32 546 CAD. If the per annum compound rate was 4.2%, how much was the loan for?

5 If you invest 5000 Swiss Francs today in an investment that pays 3.25% per annum, compounded monthly, how long will it take

TOK

Do all societies view investment and interest in the same way?

What is your stance?

Draft

for the investment to double in value, assuming no additional withdrawals or deposits are made?

6 Isabella's parents are starting a college fund for her today, on her fifth birthday. Assuming they make no additional deposits or withdrawals, if they want to give her 50 000 Brazilian Real on her eighteenth birthday, how much do they need to deposit now if the bank is offering an interest rate of 5.5%, compounded monthly. Round your answer to the nearest whole number.

7 Oliver and Harry were each given 400 GBP for their birthday. Oliver puts his in a savings account that pays 1.25% interest, compounded monthly. Harry chooses to invest in a mutual fund paying 1.75% interest, compounded annually. If each brother does not touch the money for five years, who will have earned more?

8 Ahmed has 40 000 Egyptian Pounds to invest. He wants to split his money equally between a savings account with simple interest rate of 1.2% paid annually and a two-year guaranteed investment certificates (GIC) with an interest rate of 3.5%, compounded monthly. How long will he have to leave the money in the savings account so that it earns the same amount as the GIC?

Population growth

Growth can follow different patterns and we can use these patterns to predict and compare outcomes, in order to make decisions.

Population growth is defined as the increase of the number of individuals in a population over time. It is important to study population growth and predict future trends so that decisions can be made about allocation of resources.

Population growth is based on the number of births and deaths over a certain amount of time for a given population.

Investigation 9

The population of Canada can be found in the table below:

Year	2013	2014	2015	2016	2017
Population (millions)	35.152	35.535	35.832	36.264	36.708

Source: Statistics Canada

Part 1:

1 Calculate the percentage growth rate between each year in the table.

2 Find the average percentage growth rate.

Draft

3 Use this average rate to create a general formula for the population of Canada in any given year.

4 Use your general formula to estimate the population of Canada in 2025. Is your value realistic? What factors might affect this value?

Part 2:

1 Graph the data above with time on the x-axis and population on the y-axis.

2 On the same axes, graph your general formula from question **3** above.

3 Comment on how closely your formula models the actual data.

4 **Conceptual** How can you fit a model to real-life growth data?

5 **Conceptual** How can using sequences to model growth help us make predictions and comparisons?

Students could find data for a different country or region and model in the same way as in this investigation. This could be turned into a mini-exploration.

Example 26

In 2015, the population of the Greater Tokyo Area, Japan, was 13.491 million people. In 2016, 405 000 people moved into the city while 331 000 moved out.

a Find the net increase in the population of the city.

b Calculate the annual population growth rate.

c Assuming the growth rate stays the same, find a general formula for the population.

d Estimate the population of Toyko in 2025. Is this value reasonable? Explain your answer.

Continued on next page

Draft

a $405\,000 - 331\,000 = 74\,000$	The net increase is the number of people who moved in minus the number who moved out.
b $\frac{74\,000}{13\ 491\,000} = 0.005485\ldots \approx 0.00549$	The growth rate is the net increase divided by the population.
c $u_1 = 13\,491\,000$ $r = 1.005485\ldots \approx 1.00549$ This makes the general formula: $P = 13\,491\,000\,(1.00549)^t$, where P is the population and t is number of years after 2015.	Other variables can be used for the general formula.
d $P = 13\,491\,000(1.00549)^{10}$ $P = 14\,249\,167.99967\ldots$ $P \approx 14\,249\,168$ people This is not reasonable as it does not account for any births or deaths.	

Example 27

A biologist is growing a culture of bacteria in a lab. She counts the number of bacteria in a dish every six hours. Below are her results.

Time	0	6	12	18
Number of bacteria	125	350	980	2744

a Create a general formula to model the number of bacteria at any given time.

b Use your general formula to predict the number of bacteria at the end of the second day.

c Determine when the number of bacteria will exceed 250 000.

d Will the bacteria continue to multiply indefinitely? Explain your answer.

a $u_1 = 125$ $r = \frac{350}{125} = 2.8$ This makes the general formula: $N = 125(2.8)^t$, where N is number of bacteria and t is the number of six-hour periods that have elapsed.	Other variables can be used for the general formula. Be sure to define the variables you choose.

Draft

b The end of the second day would be 48 hours. Since we have defined t as the number of six-hour periods that have elapsed:

$t = \frac{48}{6} = 8.$

$N = 125(2.8)^8$

$N = 472252.4979$

$N \approx 472\ 252$ bacteria

Round down to the nearest whole number here, as it is not possible to have part of a bacteria.

c $125(2.8)^t > 250\ 000$

By GDC:

$t > 7.3822\ldots$

Since t is every six hours:

$t > 7.3822\ldots \times 6 = 44.3$ hours

or

$t > 1.85$ days

d No. The bacteria will eventually start to die because of factors such as overcrowding.

Exercise 1M

1 A high white blood cell count can indicate that the patient is fighting an infection. A doctor is monitoring the number of white blood cells in one of her patients after receiving antibiotics. The lab returns the following data.

Hour	0	12	24	36
White blood cells (cells mcL^{-1})	12 500	11 000	9680	8518.4

a Create a general formula to model the patient's white blood cell count at any given time.

b Use your general formula to calculate the number of white blood cells this patient will have after three days.

c Discuss the limitations of your general formula.

2 The US unemployment rate for the first three months of a year is shown in the table below.

Month	January	February	March
Rate	7.9%	7.7%	7.5%

a What sort of sequence is this?

b Write a formula to calculate the unemployment rate for any month.

c Use your general formula to estimate the unemployment rate in December of the same year.

d Is it realistic to expect the unemployment rate to continue to decrease forever? Explain your answer.

3 Half-life is the time required for a substance to decay to half of its original amount.

a A radioactive isotope has a half-life of 1.23 years. Explain what this means.

b Write a general formula to calculate the amount remaining of the substance.

c Use your GDC to sketch a graph of this situation.

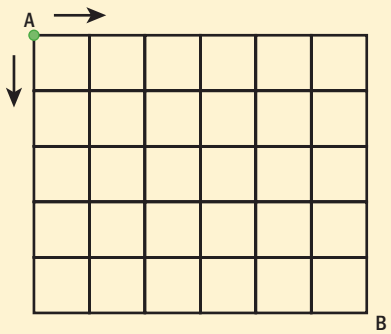
d If you start with a 52-gram sample of the isotope, how much will remain in 7.2 years?

Draft

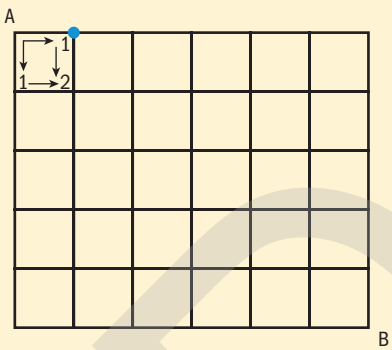
1.5 The binomial theorem

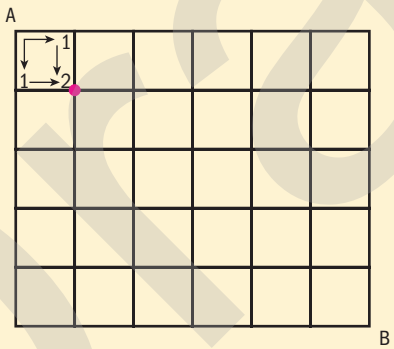
Investigation 10

A taxi cab is situated at point A and must travel to point B. It may only move right or down from any given spot, without moving back on itself. You want to figure out how many shortest possible routes there are from A to B.

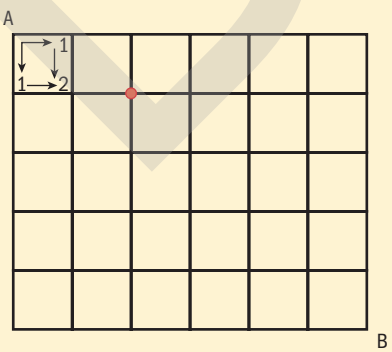


Start with the green dot. There are two possible routes because the taxi can move right and down or down and right.



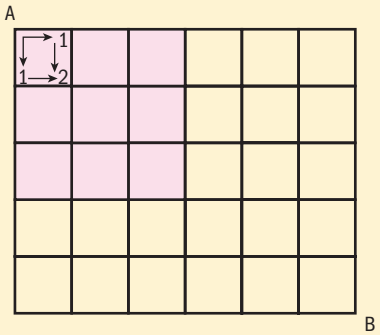


1 Can you figure out the number of shortest possible routes from A to the red dot? Mark each intersection between A and the red dot to help you.



Draft

2 Use the same notation as in the first box to fill in the shaded boxes in the grid below.



3 Turn the grid 45° so that the diagram is now a diamond with point A at the top. Describe the patterns you see in the numbers you have written so far.

4 Describe how these patterns can help you to complete the rest of the grid.

5 Complete the grid and state how many possible routes there are from A to B.

Investigation 11

The pattern you saw in the last investigation is called Pascal's triangle.

Part 1:

Blaise Pascal (1623–1662), a French mathematician, physicist and inventor, is credited with the triangular pattern below:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

1 Describe the pattern used to create each successive row of the triangle.

2 Copy the triangle above and add five more rows.

3 What do you notice about the terms on the left end and on the right end of each row?

Part 2:

1 Find the sum of each of the rows in the triangle above.

2 What do you notice about these values?

3 State the general formula to find the sum of any row of Pascal's triangle.

4 Find the sum of the 16th row.

TOK

Why do we call this Pascal's triangle when it was in use before Pascal was born?

Are mathematical theories merely the collective opinions of different mathematicians, or do such theories give us genuine knowledge of the real world?

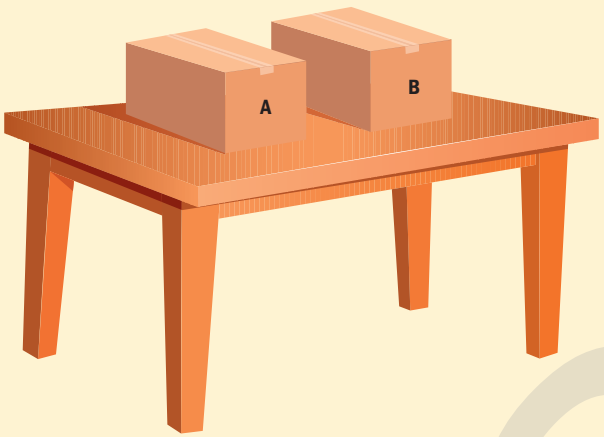
Draft

Investigation 12

Pascal's triangle is related to another branch of mathematics called combinatorics, which deals with the arrangement and combinations of objects.

Part 1:

Imagine there are two boxes on a table and you are asked to select **up to** two boxes.



You could choose to take no boxes, one box or two boxes.

If we examine the number of ways to choose the above number of boxes, we see that there is one way to choose no boxes and one way to choose both boxes. However, if we choose one box, there are two possibilities, box A or box B.

So, the number of possible combinations is:

1 2 1

1 Imagine there are three boxes, labelled A, B and C. Find the different combinations for:

a choosing no boxes **b** choosing one box
c choosing two boxes **d** choosing three boxes.

2 Total the number of combinations for each number of boxes in question **1**.

3 Repeat the process for four boxes.

4 Look back at the last investigation on Pascal's triangle.

5 What do you notice about your combinations in question **2** and **3**?

6 How can Pascal's triangle be used to calculate the number of combinations for choosing 0, 1, 2, 3, … items from a given number of items?

7 Rewrite Pascal's triangle using ${}_n\mathrm{C}_r$ notation.

The notation for a combination can be $\binom{n}{r}$, ${}^n\mathrm{C}_r$ or ${}_n\mathrm{C}_r$, where n is the number of objects available and r is the number of objects we want to choose.

8 **Conceptual** How would you use Pascal's triangle to explain why ${}_n\mathrm{C}_r = {}_n\mathrm{C}_{n-r}$?

TOK

Although Pascal is credited with bringing this triangle to the Western world, it had been known in China as early as the 13th century. What criteria should be used to determine who "invented" a mathematical discovery?

TOK

How many different tickets are possible in a lottery? What does this tell us about the ethics of selling lottery tickets to those who do not understand the implications of these large numbers?

Draft

Example 28

Calculate the following combinations using Pascal's triangle:

a ${}_4C_2$ **b** ${}^6C_5$ **c** $\binom{3}{3}$ **d** $\binom{5}{3}$

a Since $n = 4$, you need the fifth row.

1 4 6 4 1

Since $r = 2$, you need the third term.

${}_4C_2 = 6$

b ${}^6C_5$ means seventh row, sixth term.

${}_6C_5 = 6$

c $\binom{3}{3}$ means fourth row, fourth term, or you could use the fact ${}_nC_n = 1$

d $\binom{5}{3}$ means sixth row, fourth term.

$\binom{5}{3} = 10$

Another way to evaluate a combination is to use the formula:

$$ {}^nC_r = \frac{n!}{r!(n-r)!} $$

Where $n!$, called "n factorial", represents the multiplication of each preceding integer starting at n down to 1.

$$ n! = n(n-1)(n-2)(n-3) \ldots 3 \times 2 \times 1 $$

For example, $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$.

Example 29

a Calculate the following combinations using the formula without using your GDC:

i $\binom{7}{3}$ **ii** ${}_8C_5$ **iii** $\binom{4}{1}$ **iv** ${}^6C_4$

b Find the following combinations by using your GDC:

i $\binom{32}{9}$ **ii** ${}_{25}C_{12}$

Continued on next page

Draft

a i $\binom{7}{3} = \frac{7!}{3!(7-3)!}$

$\binom{7}{3} = \frac{7!}{3!4!}$

Expand each factorial.

$\binom{7}{3} = \frac{7\times6\times5\times4\times3\times2\times1}{(3\times2\times1)(4\times3\times2\times1)}$

$\binom{7}{3} = \frac{7\times6\times5\times\cancel{4\times3\times2\times1}}{(3\times2\times1)(\cancel{4\times3\times2\times1})}$

Cancel factors as appropriate.

$\binom{7}{3} = \frac{7\times\cancel{6}\times5}{\cancel{(3\times2\times1)}}$

$\binom{7}{3} = 7\times5$

Multiply the remaining factors.

$\binom{7}{3} = 35$

ii ${}_8C_5 = \frac{8!}{5!(8-5)!}$

${}_8C_5 = \frac{8!}{5!3!}$

${}_8C_5 = \frac{8\times7\times6\times5\times4\times\cancel{3\times2\times1}}{(5\times4\times3\times2\times1)(\cancel{3\times2\times1})}$

${}_8C_5 = \frac{8\times7\times\cancel{6\times5\times4}}{\cancel{5\times4\times3\times2}\times1}$

${}_8C_5 = 8\times7$

${}_8C_5 = 56$

iii $\binom{4}{1} = \frac{4!}{1!(4-1)!}$

$\binom{4}{1} = \frac{4!}{1!3!}$

$\binom{4}{1} = \frac{4\times\cancel{3\times2\times1}}{\cancel{3\times2\times1}}$

$\binom{4}{1} = 4$

Draft

iv ${}^6C_4 = \dfrac{6!}{4!(6-4)!}$

$${}^6C_4 = \frac{6!}{4!2!}$$

$${}^6C_4 = \frac{6\times5\times4\times3\times2\times1}{(4\times3\times2\times1)(2\times1)}$$

$${}^6C_4 = \frac{6\times5\times\cancel{4\times3\times2\times1}}{\cancel{(4\times3\times2\times1)}(2\times1)}$$

$${}^6C_4 = \frac{30}{2}$$

$${}^6C_4 = 15$$

b **i** 28 048 800

ii 5 200 300

Investigation 13

Recall that a **binomial** is a simplified algebraic expression of the sum or difference of two terms. For example,

$$x+3, \quad -2a+4b, \quad x^2+6, \quad x+y$$

A **binomial expansion** is the result when we expand a binomial raised to a power.

Look at the binomial expansions of $(x+y)^n$ as shown in the table below:

$(x+y)^0$	1
$(x+y)^1$	$x+y$
$(x+y)^2$	$x^2+2xy+y^2$
$(x+y)^3$	$x^3+3x^2y+3xy^2+y^3$
$(x+y)^4$	$x^4+4x^3y+6x^2y^2+4xy^3+y^4$
$(x+y)^5$	$x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$

1 What do you notice about:

- **a** the relationship between the value of n and the number of terms in the expansion
- **b** the exponents of x in each expansion
- **c** the exponents of y in each expansion
- **d** the sum of the exponents of x and y in each term of each expansion
- **e** the coefficients of the terms in each expansion?

2 Hypothesize the expansion of $(x+y)^6$.

Using these patterns is a generalization called the **binomial theorem** which provides an efficient method for expanding binomial expressions.

3 **Conceptual** How can you explain the coefficients of terms in the binomial expansion using Pascal's triangle?

4 **Conceptual** How does the binomial theorem use combinations?

Draft

Example 30

Use the binomial theorem to expand the following:

a $(x+2)^5$ **b** $(3-a)^6$ **c** $(2x+y)^4$ **d** $(3x-2y)^6$

a Setting up each term using the patterns discussed in the previous investigation:

Binomial coefficient	x	2
${}_5C_0$	x^5	2^0
${}_5C_1$	x^4	2^1
${}_5C_2$	x^3	2^2
${}_5C_3$	x^2	2^3
${}_5C_4$	x^1	2^4
${}_5C_5$	x^0	2^5

Binomial coefficient	x	2
1	x^5	1
5	x^4	2
10	x^3	4
10	x^2	8
5	x	16
1	1	32

Multiplying each term:

$(1)x^5(1) = x^5$

$(5)x^4(2) = 10x^4$

$(10)x^3(4) = 40x^3$

$(10)x^2(8) = 80x^2$

$(5)x(16) = 80x$

$(1)(1)(32) = 32$

$\therefore (x+2)^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

For the binomial coefficients, use the formula ${}_nC_r$, where n is the exponent in the expansion and r starts at 0 and increases by 1 each term.

Be careful to distinguish between the "coefficient" and the "binomial coefficient". If, for example, you were asked to find the coefficient in x^4, the answer here would be 10.

Draft

b

Binomial coefficient	3	$-a$
${}_6C_0$	3^6	$(-a)^0$
${}_6C_1$	3^5	$(-a)^1$
${}_6C_2$	3^4	$(-a)^2$
${}_6C_3$	3^3	$(-a)^3$
${}_6C_4$	3^2	$(-a)^4$
${}_6C_5$	3^1	$(-a)^5$
${}_6C_6$	3^0	$(-a)^6$

Binomial coefficient	3	$-a$
1	729	1
6	243	$-a$
15	81	a^2
20	27	$-a^3$
15	9	a^4
6	3	$-a^5$
1	1	a^6

Multiplying each term:

$(1)(729)(1) = 729$

$(6)(243)(-a) = -1458a$

$(15)(81)(a^2) = 1215a^2$

$(20)(27)(-a^3) = -540a^3$

$(15)(9)(a^4) = 135a^4$

$(6)(3)(-a^5) = -18a^5$

$(1)(1)(a^6) = a^6$

$\therefore (3 - a)^6 = 729 - 1458a + 1215a^2 - 540a^3 + 135a^4 - 18a^5 + a^6$

Note: When one of the terms in the binomial is negative, remember to use parentheses when applying the exponent.

c $\binom{4}{0}(2x)^4 y^0 + \binom{4}{1}(2x)^3 y^1 + \binom{4}{2}(2x)^2 y^2 + \binom{4}{3}(2x)^1 y^3 + \binom{4}{4}(2x)^0 y^4$

$= 16x^4 + (4)(8x^3)y + 6(4x^2)y^2 + 4(2x)y^3 + y^4$

$= 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$

Continued on next page

Draft

d $\binom{6}{0}(3x)^6(-2y)^0 = 729x^6$

$\binom{6}{1}(3x)^5(-2y)^1 = 6(243x^5)(-2y)$

$= -2916x^5y$

$\binom{6}{2}(3x)^4(-2y)^2 = 15(81x^4)(4y^2)$

$= 4860x^4y^2$

$\binom{6}{3}(3x)^3(-2y)^3 = 20(27x^3)(-8y^3)$

$= -4320x^3y^3$

$\binom{6}{4}(3x)^2(-2y)^4 = 15(9x^2)(16y^4)$

$= 2160x^2y^4$

$\binom{6}{5}(3x)^1(-2y)^5 = 6(3x)(-32y^5)$

$= -576xy^5$

$\binom{6}{6}(3x)^0(-2y)^6 = 64y^6$

$\therefore 729x^6 - 2916x^5y + 4860x^4y^2 - 4320x^3y^3 + 2160x^2y^4 - 576xy^5 + 64y^6$

Exercise 1N

Expand the following using the binomial theorem:

1 $(x+5)^4$

2 $(2-b)^5$

3 $(2x-1)^6$

4 $(4x+y)^4$

5 $(x-3y)^3$

6 $(3x+4y)^5$

When finding just a single term in an expansion, it is easier to use the binomial theorem patterns, rather than calculate the entire expansion. For any term in the expansion of $(a+b)^n$:

$$_nC_r(a)^{n-r}(b)^r$$

where n is the binomial power and r is one less than the term number, or r = term number -1.

Draft

Example 31

Find the seventh term in the expansion of $(3x - 4y)^{10}$.

For the seventh term,

$n = 10$

$r = 7 - 1 = 6$

${}_{10}C_6(3x)^4(-4y)^6$

$210(81x^4)(4096y^6)$

$69\ 672\ 960x^4y^6$

Example 32

The fifth term in the expavnsion of $(ax + by)^n$ is $560x^3y^4$. Find the values of a and b, given that a and b are integers.

Since it is the fifth term, r must be $5 - 1 = 4$, and n must be $3 + 4 = 7$.

$\therefore$ The binomial coefficient is ${}_7C_4 = 35$

$\frac{560}{35} = 16$

Since 16 is the fourth power of an integer, it must be the coefficient of y.

$\therefore 35(x)^3\left(\sqrt[4]{16}y\right)^4$

$35(x)^3(2y)^4$

This makes the original binomial $(x + 2y)^7$.

$a = 1$ and $b = 2$

Example 33

In the expansion of $\left(x + \frac{1}{x}\right)^9$, find the term in x^3.

In order for the final simplified term to contain x^3, the only possibility is: $(x)^6\left(\frac{1}{x}\right)^3 = x^3$

Since the second exponent is 3, the term number must be $3 + 1 = 4$.

Draft

Example 34

Use the binomial theorem to estimate $(1.87)^4$ to five decimal places.

$(1.87)^4 = (1 + 0.87)^4$

$(1+0.87)^4 = \binom{4}{0}1^4 0.87^0 + \binom{4}{1}1^3 0.87^1 + \binom{4}{2}1^2 0.87^2 + \binom{4}{3}1^1 0.87^3 + \binom{4}{4}1^0 0.87^4$

$(1 + 0.87)^4 = 1 + 4(0.87)^1 + 6(0.87)^2 + 4(0.87)^3 + 0.87^4$

$(1 + 0.87)^4 = 1 + 4(0.87) + 6(0.7569) + 4(0.658503) + 0.57289761$

$(1 + 0.87)^4 = 1 + 3.48 + 4.5414 + 2.634012 + 0.57289761$

$(1 + 0.87)^4 = 12.22830961$

$(1 + 0.87)^4 \approx 12.22831$

In order to use the binomial theorem, we need to create a binomial from 1.87. The easiest way is to split 1.87 into 1 + 0.87. You can use any two numbers whose sum is 1.87.

Exercise 10

1 Find the indicated term for the expansions below:

- **a** The fifth term of $(3x - 5)^{11}$
- **b** The ninth term of $(x + 6y)^{10}$
- **c** The middle term of $(2 - 3y)^6$
- **d** The constant term of $(x^4 - 3)^9$
- **e** The last term of $\left(-2x^2 - \frac{3}{x}\right)^7$

2 The values in the fourth row of Pascal's triangle are 1, 3, 3, 1.

- **a** Write down the values in the next row of the triangle.
- **b** Hence, find the term in x in $(3x - 2)^4$.

3 **a** Find the term in x^3 in the expansion of $(x - 3)^8$.

- **b** Hence, find the term in x^5 in the expansion of $-2x\,(x - 3)^8$.

4 Find the term in x^9 in the expansion of $\left(x^3 - \frac{3}{x}\right)^{11}$.

5 In the expansion of $x^7\left(\frac{x^4}{2} + \frac{k}{x^3}\right)^7$, the constant term is 168. Find the value of k.

6 The fourth term in the expansion of $(2x - k)^8$ is $-387\,072x^5$. Find the value of k.

7 In the expansion of $(a - b^2)^6$, what is the coefficient of the term in a^3b^6?

8 Use the binomial theorem to estimate $(2.52)^3$ to three decimal places.

9 **a** Expand $(x - 5)^5$.

- **b** Hence find the term in x^4 in $(2x - 1)(x - 5)^5$.

10 **a** Show that the expansions of $(3 - 2x)^4$ and $(-2x + 3)^4$ are the same.

- **b** Will $(b - ax)^n$ and $(-ax + b)^n$ always have the same expansion? Explain your answer.

11 Find the binomial power, in the form $(a + b)^n$, with expansion $27x^3 - 108x^2y + 144xy^2 - 64y^3$.

12 Show that ${}_nC_0 + {}_nC_1 + {}_nC_2 + \ldots {}_nC_{n-1} + {}_nC_n = 2^n$.

Developing your toolkit

Now do the Modelling and investigation activity on page 60.

Draft

1.6 Proofs

Investigation 14

How do we know that $3(x-2)-4(3x+5)+2x-2=-7(x+4)$ is true for all values of x?

1 Using substitution, show the statement holds true for $x=1$.

2 Using substitution, show the statement holds true for $x=-2$.

3 **Factual** If the proof holds true for two values of x, does this mean it will hold true for **all** values of x?

4 Draw a vertical line down the middle of your page, to create two columns. At the top, label the left-hand side column with "LHS" and the right-hand side column with "RHS". Write the LHS side of the statement above under the first column and the RHS under the second column.

Use algebra to simplify the LHS. When it is fully simplified, think of a way to make it look exactly like the RHS.

5 **Conceptual** Explain why using algebra is a better method than using substitution.

A mathematical **proof** is a series of logical steps that show one side of a mathematical statement is equivalent to the other side for all values of the variable.

We need proofs in mathematics to show that the mathematics we use every day is correct, logical and sound.

There are many different types of proofs (deductive, inductive, by contradiction), but we will use **algebraic proofs** in this section.

The goal of an algebraic proof is to transform one side of the mathematical statement until it looks exactly like the other side.

One rule is that you **cannot** move terms from one side to the other. Imagine that there is an imaginary fence between the two sides of the statement and terms cannot jump the fence!

At the end of a proof, we write a concluding statement, such as LHS ≡ RHS or QED.

TOK

"Mathematics may be defined as the economy of counting. There is no problem in the whole of mathematics which cannot be solved by direct counting."

—E. Mach

To what extent do you agree with this quote?

EXAM HINT

QED is an abbreviation for the Latin phrase "quod erat demonstrandum" which means "what was to be demonstrated" or "what was to be shown". The sign with three bars (≡) is the identity symbol, which means that the two sides are equal by definition.

Draft

Example 35

Prove that $\frac{6}{8} = \frac{1}{4} + \frac{1}{2}$.

LHS	RHS
$\frac{6}{8}$	$\frac{1}{4} + \frac{1}{2}$ $\frac{1}{4} + \frac{2}{4}$ $\frac{3}{4}$ $\frac{3}{4} \times \frac{2}{2}$ $\frac{6}{8}$ QED

First, simplify the RHS using a common denominator.

Once simplified completely, in order for the RHS to match the LHS, multiply by $\frac{2}{2}$ (which is equivalent to 1).

Don't forget a concluding statement at the end.

Exercise 1P

1 Prove that $-2(a-4) + 3(2a+6) - 6(a-5) = -2(a-28)$.

2 Prove that $(x-3)^2 + 5 = x^2 - 6x + 14$.

3 Prove that $\frac{1}{m} = \frac{1}{m+1} + \frac{1}{m^2+m}$.

4 **a** Prove that $\frac{x-2}{x} \div \frac{3x-6}{x^2+x} = \frac{x+1}{3}$.

b For what values of x does this mathematical statement **not** hold true?

TOK

What is the role of the mathematical community in determining the validity of a mathematical proof?

Chapter summary

- A **sequence** (also called a **progression**) is a list of numbers written in a particular order. Each number in a sequence is called a **term**.
- A **finite sequence** has a fixed number of terms.
  An **infinite sequence** has an infinite number of terms.
- A formula or expression that mathematically describes the pattern of the sequence can be found for the general term, u_n.
- A sequence is called **arithmetic** when the same value is added to each term to get the next term.
- A sequence is called **geometric** when each term is multiplied by the same value to get the next term.
- A **recursive sequence** uses the previous term or terms to find the next term. The general term will include the notation u_{n-1}, which means "the previous term".
- A **series** is created when the terms of a sequence are added together.
  A sequence or series can be either finite or infinite.
  A **finite series** has a fixed number of terms.
  For example, $7 + 5 + 3 + 1 + -1 + -3$ is finite because it ends after the sixth term.
  An **infinite series** continues indefinitely.
  For example, the series $10 + 8 + 6 + 4 + \ldots$ is infinite because the ellipsis indicates that the series continues indefinitely.

Draft

- The formula for any term in an arithmetic sequence is: $u_n = u_1 + (n-1)d$
- The formula for any term in a geometric sequence is: $u_n = u_1 r^{n-1}$.
- The sum, S_n, of an arithmetic series can be expressed as: $S_n = \frac{n}{2}(u_1 + u_n)$.
- The sum, S_n, of an arithmetic series can also be expressed as: $S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$
- For geometric series, the formula for the sum is:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r},\ r \neq 1$$

- The sum of a converging infinite geometric series is: $S_\infty = \frac{u_1}{1-r}, |r| < 1$
- **Simple interest** is interest paid on the initial amount borrowed, saved or invested (called the **principal**) only and not on past interest.

  Compound interest is paid on the principal and the accumulated interest. Interest paid on interest!
- We can calculate combinations using the formula ${}^nC_r = \frac{n!}{r!(n-r)!}$
- A mathematical **proof** is a series of logical steps that show one side of a mathematical statement is equivalent to the other side for all values of the variable.
- We need proofs in mathematics to show that the mathematics we use every day is correct, logical and sound.

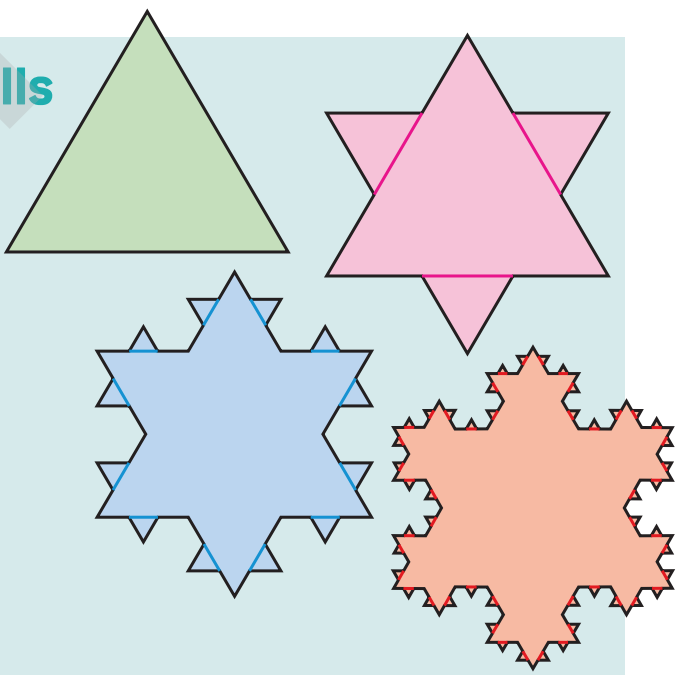
  The goal of an algebraic proof is to transform one side of the mathematical statement until it looks exactly like the other side.

  One rule is that you **cannot** move terms from one side to the other.

  At the end of a proof, we write a concluding statement, such as LHS ≡ RHS or QED.

Developing inquiry skills

Return to the opening problem. How has your understanding of the Koch snowflake changed as you have worked through this chapter? What features of, for example, the ninth iteration can you now work out from what you have learned?



Draft

Chapter review

Click here for a mixed review exercise

1 For each of the following sequences,

a Identify whether the sequence is arithmetic, geometric or neither.

b If it is arithmetic or geometric, find an expression for u_n.

c If it is arithmetic or geometric, find the indicated term.

d If it is arithmetic or geometric, find the indicated sum.

i 3, 6, 18, ... u_8, S_{12}

ii $-16, -14, -12, \ldots$ u_{10}, S_8

iii 2000, 1000, 500, ... u_9, S_7

iv $\sum_{n=1}^{\infty} 3 \times 2^{n-1}$, u_5, S_{10}

v The consecutive multiples of 5 greater than 104, u_7, S_9

2 In an arithmetic sequence, $u_6 = -5$ and $u_9 = -20$. Find S_{20}.

3 Write down the first five terms for the recursive sequence $u_n = -2u_{n-1} + 3$ with $u_1 = -4$.

4 For the geometric series $0.5 - 0.1 + 0.02 \ldots$ $S_n = 0.416$. Find the number of terms in the series.

5 For a geometric progression, $u_3 = 4.5$ and $u_7 = 22.78125$. Find the value of the common ratio and the first term.

6 Which of the following sequences has an infinite sum? Justify your choice and find that sum.

A $\frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$ **B** 0.06, 0.12, 0.24, ...

7 How many terms are in the sequence 4, 7, 10, ..., 61?

8 In a geometric sequence, the fourth term is 8 times the first term. If the sum of the first 10 terms is 765, find the 10th term of the sequence.

9 Three consecutive terms of a geometric sequence are $x - 3$, 6 and $x + 2$. Find all possible values of x.

10 A tank contains 55 litres of water. Water flows out at a rate of 7% per minute.

a Write a sequence that represents the volume left in the tank after 1 minute, 2 minutes, 3 minutes, etc.

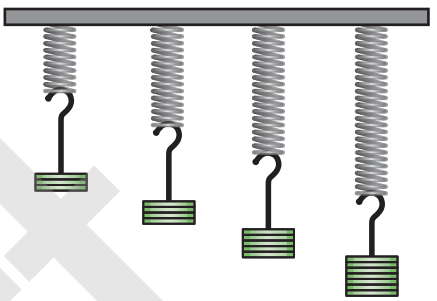
b What kind of sequence did you write? Justify your answer.

c How much water is left in the tank after 10 minutes?

d How much water has been drained after 15 minutes?

e How long will it take to drain the tank?

11 Marie-Jeanne is experimenting with weights of 100 g attached to a spring.



She records the weight and length of the spring after attaching the weight.

Number of 100 g weights attached to the spring	1	2	3	4
Length of spring (cm)	45	49	53	57

a Write a general formula that represents the length of the spring if n weights are attached.

b Calculate the supposed length of a spring if a 1 kg weight is attached.

c Explain any limitations of Marie-Jeanne's experiment.

d If the spring stretches to 101 cm, what is the total weight that was attached to the spring?

12 Kostas gets a four-year bank loan to buy a new car that is priced at €20 987. After the four years, Kostas will have paid the bank a total of €22 960. What annual interest rate did the bank give him if the interest was compounded monthly?

13 The first seven numbers in row 14 of Pascal's triangle are 1, 13, 78, 286, 715, 1287, 1716.

a Complete the row and explain your strategy.

b Explain how you can use your answer in part **a** to find the 15th row. State the terms in the 15th row.

14 Using the binomial theorem, expand $(3x - y)^6$.

15 Find the coefficient of the term in x^2 in the expansion of $\left(\frac{3}{x^2} - 4x^4\right)^8$.

16 In the binomial expansion of $\left(\frac{2}{x^2} - 5x^5\right)^n$, the sixth term contains x^{25}. Solve for n.

Draft

17 **a** Find the term in x^5 in the expansion of $(x-3)^9$.

b Hence, find the term in x^6 in the expansion of $-2x(x-3)^9$.

18 In the expansion of $\left(\frac{x^3}{3}+\frac{k}{x}\right)^{12}$, the constant is $\frac{112\,640}{27}$. Find the value of k.

19 Prove that:

$(2x-1)(x-3)-3(x-4)^2 = -x^2-31x+51$.

20 **a** Prove that $\frac{x^2-x-6}{x+4}\cdot\frac{x^2-16}{x^2+2}=\frac{x^2-7x+12}{x}$.

b For what values of x does this mathematical statement **not** hold true?

Exam-style questions

21 **P1:** **a** Find the binomial expansion of $\left(1-\frac{x}{4}\right)^5$ in ascending powers of x. (3 marks)

b Using the first three terms from the above expansion, find an approximation for 0.975^5. (3 marks)

22 **P1:** The 15th term of an arithmetic series is 143 and the 31st term is 183.

a Find the first term and the common difference. (5 marks)

b Find the 100th term of the series. (2 marks)

23 **P2:** Angelina deposits \$3000 in a savings account on 1 January 2019, earning compound interest of 1.5% per year.

a Calculate how much interest (to the nearest dollar) Angelina would earn after 10 years if she leaves the money alone. (3 marks)

b In addition to the \$3000 deposited on January 1st 2019, Angelina deposits a further amount of \$1200 into the same account on an annual basis, beginning on 1st January 2020. Calculate the total amount of money in her account at the start of January 2030 (before she has deposited her money for that year). (4 marks)

24 **P2:** Brad deposits \$5500 in a savings account which earns 2.75% compound interest per year.

a Determine how much Brad's investment will be worth after 4 years. (3 marks)

b Calculate, to the nearest year, how long Brad must wait for the value of the investment to reach \$12 000. (5 marks)

25 **P2:** Find the coefficient of the term in x^5 in the binomial expansion of $(3+x)(4+2x)^8$. (4 marks)

26 **P1:** The coefficient of x^2 in the binomial expansion of $(1+3x)^n$ is 495. Determine the value of n. (6 marks)

27 **P2:** Find the constant term in the expansion of $\left(x^3-\frac{2}{x}\right)^8$. (4 marks)

28 **P2:** **a** Find the binomial expansion of $\left(\frac{1}{2x}-x\right)^4$ in ascending powers of x. (3 marks)

b Hence, or otherwise, find the term independent of x in the binomial expansion of $(3-x)^3\left(\frac{1}{2x}-x\right)^4$. (4 marks)

29 **P2:** A convergent geometric series has sum to infinity of 120.

Find the 6th term in the series, given that the common ratio is 0.2. (5 marks)

30 **P1:** The second term in a geometric series is 180 and the sixth term is $\frac{20}{9}$.

Find the sum to infinity of the series. (7 marks)

31 **P2:** Find the value of $\sum_{n=0}^{n=15}\left(1.6^n-12n+1\right)$, giving your answer correct to 1 decimal place. (6 marks)

32 **P1:** A ball is dropped from a vertical height of 20 m.

Following each bounce, it rebounds to a vertical height of $\frac{5}{6}$ its previous height.

Assuming that the ball continues to bounce indefinitely, show that the maximum distance it can travel is 220 m. (5 marks)

33 **P1:** Prove the binomial coefficient identity $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$. (6 marks)

34 **P2:** Find the sum of all integers between 500 and 1400 (inclusive) that are not divisible by 7. (7 marks)

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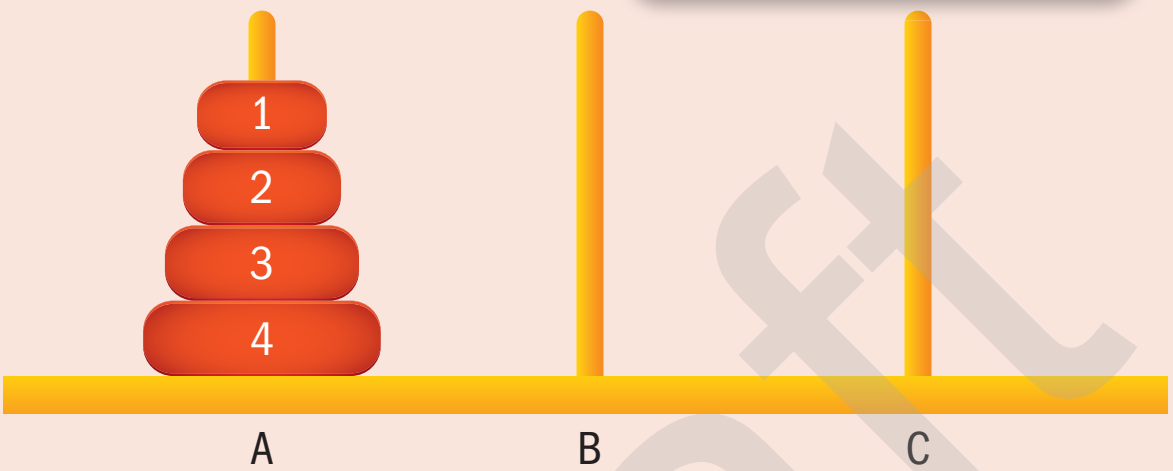
The Towers of Hanoi

Approaches to learning: Thinking skills, Communicating, Research

Exploration criteria: Mathematical communication (B), Personal engagement (C), Use of mathematics (E)

IB topic: Sequences

The problem



The aim of the **Towers of Hanoi problem** is to move all the disks from peg A to peg C following these rules:

1. Move only one disk at a time.
2. A larger disk may not be placed on top of a smaller disk.
3. All disks, except the one being moved, must be on a peg.

For 64 disks, what is the minimum number of moves needed to complete the problem?

Explore the problem

Use an online simulation to explore the Towers of Hanoi problem for three and four disks.

What is the minimum number of moves needed in each case?

Solving the problem for 64 disks would be very time consuming, so you need to look for a rule for n disks that you can then apply to the problem with 64 disks.

Try and test a rule

Assume the minimum number of moves follows an arithmetic sequence.

Use the minimum number of moves for three and four disks to predict the minimum number of moves for five disks.

Check your prediction using the simulator.

Does the minimum number of moves follow an arithmetic sequence?

Find more results

Use the simulator to write down the number of moves when $n = 1$ and $n = 2$.

Organize your results so far in a table.

Look for a pattern. If necessary, extend your table to more values of n.

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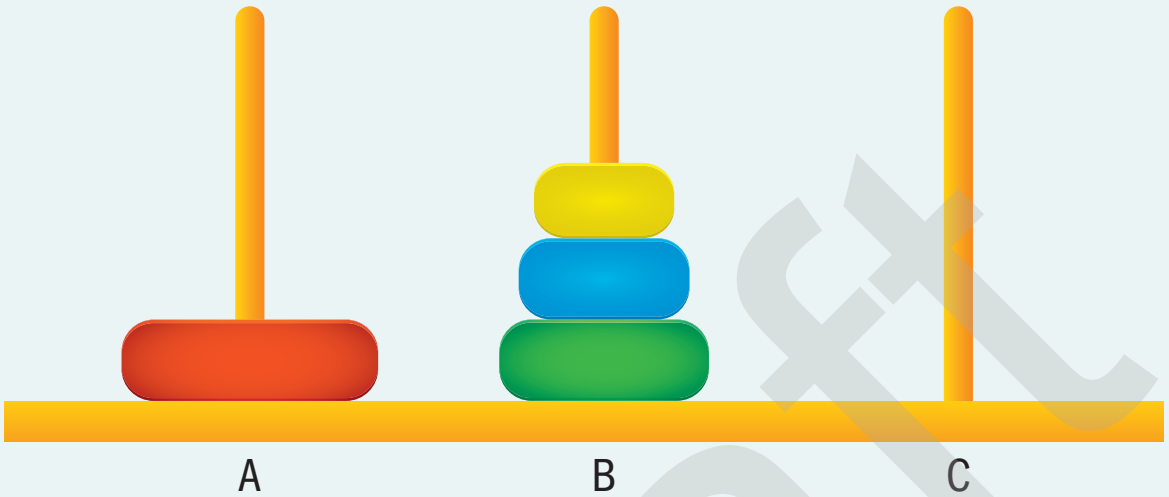
Try a formula

Return to the problem with four disks.

Consider this image of a partial solution to the problem. The large disk on peg A has not yet been moved.



Consider your previous answers.

What is the minimum possible number of moves made so far?

How many moves would it then take to move the largest disk from peg A to peg C?

When the large disk is on peg C, how many moves would it then take to move the three smaller disks from peg B to peg C?

How many total moves are therefore needed to complete this puzzle?

Use your answers to these questions to write a formula for the minimum number of moves needed to complete this puzzle with n disks.

This is an example of a **recursive formula**. What does that mean?

How can you check if your recursive formula works?

What is the problem with a recursive formula?

Try another formula

You can also try to solve the problem by finding an **explicit formula** that does not depend on you already knowing the previous minimum number.

You already know that the relationship is not arithmetic.

How can you tell that the relationship is not geometric?

Look for a pattern for the minimum number of moves
in the table you constructed previously.

Hence write down a formula for the minimum number of moves in terms of n.

Use your explicit formula to solve the problem with 64 disks.

Extension

- What would a solution look like for four pegs? Does the problem become harder or easier?
- Research the "Bicolor" and "Magnetic" versions of the Towers of Hanoi puzzle.
- Can you find an explicit formula for other recursive formulae? (eg Fibonacci)

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