1 From patterns to generalizations: sequences, series and proof

One does not have to look far and wide to find visual patterns, but some questions immediately arise.

Concepts

- Patterns
- Generalizations

Microconcepts

- Arithmetic and geometric sequences and series
- Introduction to limits
- Sum of series
- Permutations and combinations
- Proof
- Binomial theorem

Can these patterns be explained mathematically?

Can patterns in numbers be useful in real-life situations?

What information would you require to choose the best loan offer? What other scenarios could this be applied to?

If you take out a loan to buy a car how can you determine the actual amount it will cost?

How can a finite area be enclosed by an infinite perimeter?

The diagrams here are the first four iterations to create a Koch snowflake.

Describe what you observe about:

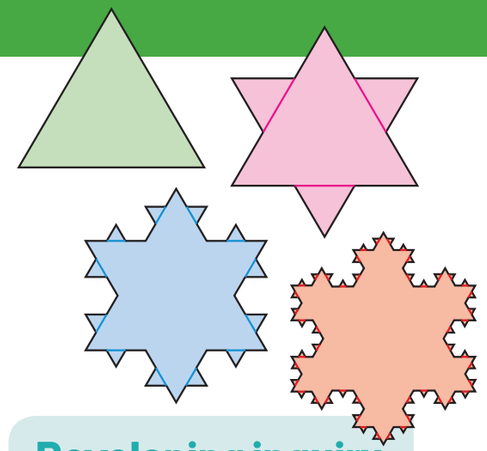
- the perimeter as you move from the equilateral triangle through to the fourth iteration
- the area enclosed as you move from the triangle to the fourth iteration.

What changes would you expect in the fifth iteration?

How would you physically measure the perimeter at the fifth iteration if the original triangle had sides of 1m in length?

What happens if you start with a square instead of an equilateral triangle?

Can a finite volume enclose an infinite area?



Developing inquiry skills

To come

Before you start

Click here for help with this skills check

You should know how to:

1 Solve linear algebraic equations.
eg: $x - 3(x+5) = 20 - 3x$

$\Rightarrow x - 3x - 15 = 20 - 3x$

$\Rightarrow -2x - 15 = 20 - 3x$

$\Rightarrow x = 35$

2 Simplify surds.

eg simplify $\dfrac{\sqrt{2}}{1-\sqrt{2}}$

$$\frac{\sqrt{2}}{1-\sqrt{2}} = \frac{\sqrt{2}\left(1+\sqrt{2}\right)}{\left(1-\sqrt{2}\right)\left(1+\sqrt{2}\right)} = \frac{\sqrt{2}+2}{1-2} = -2-\sqrt{2}$$

3 Manipulate algebraic fractions.

eg simplify $\dfrac{x+3}{x} = \dfrac{2}{x+1} - \dfrac{3x}{x-1}$

$$= \frac{(x+3)(x+1)(x-1)+2x(x-1)-3x^2(x+1)}{x(x+1)(x-1)}$$

$$= \frac{(x+3)(x^2-1)+2x^2-2x-3x^3-3x^2}{x(x^2-1)}$$

$$= \frac{x^3-x+3x^2-3+2x^2-2x-3x^3-3x^2}{x(x^2-1)}$$

$$= \frac{-2x^3+2x^2-3x-3}{x(x^2-1)}$$

Skills check

1 Solve the following equations:

a $3x + 5(x-4) = 20x + 4$

b $\dfrac{x+1}{2x-1} = \dfrac{x-3}{2x+1}$

2 Simplify the following:

a $\dfrac{1+\sqrt{2}}{1-\sqrt{2}}$

b $\dfrac{2\sqrt{2}}{1-\sqrt{3}}$

3 Simplify:

$\dfrac{x}{x+1} - \dfrac{1}{2x-1} + \dfrac{2}{x-1}$

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1.1 Sequences, series and sigma notation

Opening investigations

You are going to start this chapter by doing some simple arithmetic with the aim of recognizing patterns. The challenge is for you to understand and explain the patterns that emerge. In Investigation 2, you will be asked to propose a conjecture, which is a rule generalizing findings based on observed patterns.

Investigation 1

Work out the following products:

1×1 11×11 111×111 1111×1111

1 What pattern do you see emerging?

2 Does this continue as you make the string of 1's longer?

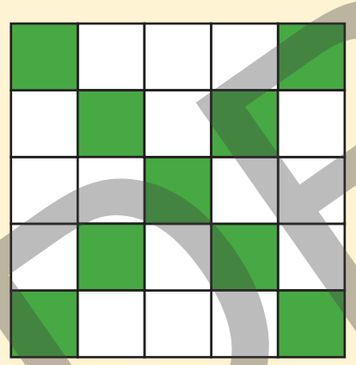
3 Can you predict when this pattern stops and explain why this happens?

International-mindedness

Where did numbers come from?

Investigation 2

This diagram represents the floor of a room covered with square tiles. It has a total of nine tiles along the main diagonals (shaded), and five tiles on each side. 25 tiles are used to cover the floor completely.



Another room has a total of 13 square tiles along the diagonals.

1 How many square tiles are there on each side in this other room?

2 How many tiles are needed to completely cover the floor?

3 What if the total number of tiles along the diagonals is 15?

4 What if there is a total of 135 tiles along the diagonals?

5 What if the total number of squares along the diagonals is an even number?

6 Continue to generate data to help you form a conjecture. Can you explain why this rule holds true?

7 How can you write the generalization concisely?

8 **Conceptual** Why is an algebraic expression more useful than generating numerical values?

A **sequence** is a list of numbers that is written in a defined order, ascending or descending, following a specific rule. Each of the numbers making up a sequence is called a **term** of that sequence. Sometimes a sequence is also referred to as a **progression**.

Look at the following sequences of numbers and identify the rule which would help you obtain the next term.

i 7, 5, 3, 1, …

ii 2, 4, 8, 16, …

iii 1, 3, 9, 27, …

Sequences may be **finite** or **infinite**.

The sequence 7, 5, 3, 1, −1, −3 is a finite sequence with six terms, whereas the sequence 7, 5, 3, 1, −1, −3, … is an infinite sequence with an infinite number of terms. The distinction is indicated by the ellipsis (…) at the end of the sequence.

A sequence is sometimes written in terms of the general term as $\{u_r\}$, where r can take values 1, 2, 3, …
If the sequence is finite then r will terminate at some point.

The sequence $\{u_r\} = \{3r - 1\}$, where $r \in \mathbb{Z}^+$ represents the infinite sequence 2, 5, 8, 11, …, whereas the sequence $\{u_r\} = \left\{\frac{1}{r^2}\right\}$, where $r \in \mathbb{Z}^+$, $r \leq 5$ represents the finite sequence $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}$.

All the terms in a sequence added together are called a **series**. Like sequences, series can be finite or infinite.

The series obtained by adding the six terms of the sequence 7, 5, 3, 1, −1, −3 is $7 + 5 + 3 + 1 - 1 - 3 = 12$. This is a finite series. The sum $1 + 3 + 9 + 27 + 81 + \ldots$ continues indefinitely and is an infinite series.

The set of positive integers $\mathbb{Z}^+$ can be written as $\{1, 2, 3, 4, 5, \ldots, r, \ldots\}$ where the letter r is used to represent the general term. If the positive integers which are multiples of 5 are considered, then the set $\{5, 10, 15, 20, \ldots 5r, \ldots\}$ is obtained. In this case the general term is $5r$ where r is any positive integer. The **harmonic series** is the infinite sum of the reciprocals of positive integers, ie $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{r} + \ldots$

Series can be represented in compact form using sigma (Σ) notation. This makes use of the general term written in terms of r, which often represents a positive integer.

The sum of the first 10 positive integers can be written as follows using sigma notation:

$$\sum_{r=1}^{10} r$$

The largest value that r can take

The smallest value that r can take

Read this as "The sum of r, from $r = 1$ to $r = 10$."

If you want to write the sum of the positive multiples of 5 less than 100, then you first need to think of the general term, which is $5r$, and then establish the range of values that r can take. The smallest positive

HINT

$\{u_r\}$ represents the sequence whereas u_r represents the rth term.

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Do the names that we give things impact how we understand them?

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multiple of 5 is 5 in which case $r = 1$, and since you want the largest multiple of 5 to be 100 the largest value that r can take is 20 because $100 = 5 \times 20$.

$$5+10+15+\ldots+100=\sum_{r=1}^{20}5r$$

Sometimes you will also have to interpret a sum given in sigma notation and expand it into individual terms. For example:

$$\sum_{r=0}^{4}(2r+1)=(2\times0+1)+(2\times1+1)+(2\times2+1)+(2\times3+1)+(2\times4+1)=1+3+5+7+9$$

HINT
In this case the series starts $r = 0$.

In Example 1 you will learn how to look for a pattern and write the general term.

Example 1

For each of the following sequences, write the next three terms and find the general term:

a 2, 7, 12, 17, … **b** 2, 6, 12, 20, … **c** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ **d** 5, 10, 20, 40, …

a The next three terms of this sequence are 22, 27, 32. The sequence can be written as: 2, 2 + 5, 2 + 10, 2 + 15 = 2, 2 + (1 × 5), 2 + (2 × 5), 2 + (3 × 5), …, 2 + (r − 1) × 5 The general term is $2 + (r-1) \times 5 = 5r - 3$, where r can take the values 1, 2, 3, …	Note that at each step we add 5 to get the next term. Write the sequence using pattern noticed.
b The next three terms are 30, 42, 56. The sequence can be written as 1 × 2, 2 × 3, 3 × 4, 4 × 5, …, $r \times (r+1)$, … The general term is $r \times (r+1)$, where r can take the values 1, 2, 3, …	Note that the given terms can be written as: 1 × 2, 2 × 3, 3 × 4, 4 × 5, …
c The next three terms are $\frac{5}{6}, \frac{6}{7}, \frac{7}{8}$. The sequence can be written as: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{r}{r+1}, \ldots$ The general term is $\frac{r}{r+1}$, where r can take the values 1,2,3, …	The pattern here is easy to follow.
d The next three terms are 80, 160, 320. The general term is $5 \times 2^{r-1}$, where r can take the values 1, 2, 3, …	Each term is obtained by multiplying the previous term by 2.

HINT
You can check the answers by putting $r = 5,6,7$ in the general term obtained in each case.

Example 2 shows how to find the terms of a sequence represented by its general term.

Example 2

Write down the first three terms of each of the following sequences:

a $\{u_r\} = \{5r - 2\}, r \in \mathbb{Z}^+$ **b** $\{u_r\} = \left\{\frac{(-1)^r}{r^2}\right\}, r \in \mathbb{Z}^+$

a $u_1 = 5 \times 1 - 2 = 3$ $u_2 = 5 \times 2 - 2 = 8$ $u_3 = 5 \times 3 - 2 = 13$ 3, 8, 13	Substitute values 1, 2 and 3 for r.
b $u_1 = \frac{(-1)^1}{1^2} = -1$ $u_2 = \frac{(-1)^2}{2^2} = \frac{1}{4}$ $u_3 = \frac{(-1)^3}{3^2} = -\frac{1}{9}$ $-1, \frac{1}{4}, -\frac{1}{9}$	Substitute values 1, 2 and 3 for r.

Example 3 shows how to represent a given sequence by its general term after recognizing a pattern.

Example 3

Write each of the following sequences using the general term:

a 3, 6, 9, 12, … **b** 2, −10, 50, −250 **c** $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots$

a 3, 6, 9, 12, … $\{u_r\} = \{3r\}, r \in \mathbb{Z}^+$	This is an infinite sequence of the positive multiples of 3.
b 2, −10, 50, −250 $\{u_r\} = \{2(-5)^{r-1}\}, r \in \mathbb{Z}^+, r \le 4$	This finite sequence can be written as: $2, 2\times(-5), 2\times25, 2\times(-125)$ which can be rewritten in terms of powers of −5: $= 2\times(-5)^0, 2\times(-5)^1, 2\times(-5)^2, 2\times(-5)^3$
c $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots$ $\{u_r\} = \left\{\frac{r}{2r+1}\right\}, r \in \mathbb{Z}^+$	In this infinite sequence, the numerators are the positive integers and the denominators are successive odd integers greater than 1.

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Example 4 shows how to expand a series written in sigma notation.

Example 4

For each of the following series written in sigma notation, write the first five terms:

a $\sum_{r=1}^{10} r(r-1)$ **b** $\sum_{r=1}^{\infty} (-1)^r r^2$ **c** $\sum_{r=1}^{\infty} \frac{r+1}{2r-1}$

a $\sum_{r=1}^{10} r(r-1) = 1\times 0 + 2\times 1 + 3\times 2 + 4\times 3 + 5\times 4 + \ldots$ $= 0 + 2 + 6 + 12 + 20 + \ldots + 90$	Substitute $r = 1$ to 5 for the first through to the fifth term. Simplify.
b $\sum_{r=1}^{\infty} (-1)^r r^2$ $= (-1)^1 \times 1^2 + (-1)^2 \times 2^2 + (-1)^3 \times 3^2 + (-1)^4 \times 4^2 + (-1)^5 \times 5^2 + \ldots$ $= -1 + 4 - 9 + 16 - 25 + \ldots$	
c $\sum_{r=1}^{\infty} \frac{r+1}{2r-1} = \frac{1+1}{2-1} + \frac{2+1}{4-1} + \frac{3+1}{6-1} + \frac{4+1}{8-1} + \frac{5+1}{10-1}$ $= 2 + 1 + \frac{4}{5} + \frac{5}{7} + \frac{6}{9} + \ldots$	

In Example 5 you will see how a given series can be written in sigma notation.

TOK

Is mathematics a language?

Example 5

Write each of the following series in sigma notation:

a $3 + 11 + 19 + 27 + 35$ **b** $1 - 1 + 1 - 1 + 1 - 1 + \ldots$ **c** $-6 + 12 - 24 + 48 - 96 + 192$

a $3 + 11 + 19 + 27 + 35$ $= \sum_{r=1}^{5} 8r - 5$	This is a finite series which can be written as: $3 + (3 + 8) + (3 + 16) + (3 + 24) + (3 + 32)$ $= 3 + (3 + 1 \times 8) + (3 + 2 \times 8) + (3 + 3 \times 8) + (3 + 4 \times 8)$ The general term is $3 + (r - 1) \times 8 = 8r - 5$.
b $1 - 1 + 1 - 1 + 1 - 1 + \ldots$ $= \sum_{r=1}^{\infty} (-1)^{r-1}$	This is an infinite series. Each term oscillates between -1 and $+1$ and the general term is $(-1)^{r-1}$.
c $-6 + 12 - 24 + 48 - 96 + 192$ $\sum_{r=1}^{6} (-1)^r 6r$	This is a finite series with oscillating signs and each term is the next multiple of 6.

Exercise 1A

1 For each of the following sequences, write the next three terms and find the general term:

a 3, 4.5, 6, 7.5, …

b 17, 14, 11, 8, …

c 3, 9, 27, 81, …

d $\frac{1}{4}, \frac{4}{7}, \frac{7}{10}, \frac{10}{13}, \ldots$

e $\frac{1}{2}, \frac{1}{12}, \frac{1}{30}, \frac{1}{42}, \ldots$

2 Write down the first five terms of each of the following sequences:

a $u_r = 3 - 2r$

b $u_r = \frac{r}{2r+1}$

c $u_r = 2r + (-1)^r r$

d $u_r = (-1)^r \times 2$

e $u_r = \frac{3}{2^{r-1}}$

3 Write each of the following sequences using the general term:

a 5, 10, 15, 20, …

b 6, 14, 22, 30, …

c $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

d $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots$

e 0, 3, 8, 15, …

4 Write each of the following series in full:

a $\sum_{r=1}^{4} 2r(1-r)$

b $\sum_{r=0}^{5} (-1)^r r^2$

c $\sum_{r=1}^{5} \frac{r}{3r-1}$

d $\sum_{r=1}^{4} 5$

e $\sum_{r=0}^{3} r^2 - 3$

5 For each of the following series written in sigma notation, write the first five terms:

a $\sum_{r=1}^{\infty} \frac{r+1}{r^2}$

b $\sum_{r=1}^{\infty} \frac{(-1)^r}{2r^2-1}$

c $\sum_{r=1}^{20} r(5r-1)$

d $\sum_{r=0}^{5} 2^r - 3$

e $\sum_{r=1}^{\infty} r^r$

6 Write each of the following series in sigma notation:

a $8 + 5 + 2 - 1 - 4$

b $3 + 10 + 21 + 36 + 55$

c $0 + \frac{1}{3} + \frac{1}{2} + \frac{3}{5} + \frac{2}{3} + \frac{5}{7} + \ldots$

d $1 + 9 + 25 + 49 + 81$

e $3k + 6k + 9k + 12k + 15k$

Developing inquiry skills

Now go back to the opening question. Suppose the length of each side of the first triangle is 81 cm. Can you work out the length of each side of the figure in each iteration? Tabulate your results and try to find a pattern and then make a conjecture.

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1.2 Arithmetic and geometric sequences and series

Investigation 3

Whenever you go through airport security you have to place your hand luggage, coat, phone, etc into a tray that goes on a conveyer belt which then takes it through an x-ray scanner.

When answering the following questions, you can assume the following:

- Trays are placed on the conveyer belt with no gaps between them.
- The length of each tray is 60 cm.
- The conveyer belt is moving at 10 cm per second.
- Each person uses three trays.

1 Copy and complete the following table:

Number of people ahead of you	Distance of your first tray to machine, d (m)	Waiting time, T (s)
0	0	0
1	1.8	
2		36
. . . .	. . . .	. . . .
n		

2 What patterns do you see emerging?

3 Now assume that there is a 30 cm gap separating trays belonging to different passengers. Construct and complete a table similar to the one above.

4 How have the patterns changed?

5 What happens if the distance between the trays of individual passengers changes to 50 cm? 60 cm? 80 cm?

6 How have the patterns changed?

7 Factual What do you notice about consecutive terms in the second and third columns?

8 Factual How would you generalize the relationship between the distance from the machine to your first tray and the number of people ahead of you?

9 Factual Write down the relationship between the waiting time and the number of people ahead of you.

10 Conceptual How did the recognition of patterns help you make predictions about waiting times?

Arithmetic sequences and series

A growth pattern that is represented by a **linear relationship** is also known as an arithmetic sequence, which is defined as follows:

If the difference between two consecutive numbers in a sequence is constant then it is an **arithmetic sequence** or an **arithmetic progression**. The constant difference is called the **common difference** and is denoted by d.

Consider how an arithmetic sequence with first term u_1 and common difference d grows:

First term u_1

Second term $u_2 = u_1 + d$

Third term $u_3 = u_2 + d = u_1 + 2d$

Fourth term $u_4 = u_3 + d = u_1 + 3d$

This leads to the general term $u_n = u_1 + (n-1)d$.

An arithmetic sequence with first term u_1 and common difference d has **general term** $u_n = u_1 + (n-1)d$.

HINT

A recursive equation is one in which the next term is defined as a function of earlier terms. In the case of an arithmetic sequence the recursive equation is $u_n = u_{(n-1)} + d$.

The next four examples show you how to use the general term formula to answer different types of questions.

Example 6

The fourth term of an arithmetic sequence is 18 and the common difference is –5. Determine the first term and the nth term.

$u_4 = u_1 + 3 \times (-5) = 18$ $\Rightarrow u_1 = 18 + 15 = 33$ $u_n = 33 + (n-1) \times (-5)$ $\Rightarrow u_n = 38 - 5n$	Using $u_n = u_1 + (n-1)$d.

Example 7

Find the number of terms in the following arithmetic sequences:

a 20, 23, 26, …, 83 **b** 34, 30, 26, …, –30 **c** $6a, 4a, 2a, \ldots, -22a$

a $u_1 = 20,\ d = 3$ $u_n = 17 + 3n = 83$ $\Rightarrow n = 22$	$d = 23 - 20 = 3$ Using $u_n = u_1 + (n-1)d$. Solve the linear equation to obtain n.
b $u_1 = 34,\ d = -4$ $u_n = 38 - 4n = -30$ $\Rightarrow n = 17$	
c $u_1 = 6a,\ d = -2a$ $u_n = 8a - 2an = -22a$ $\Rightarrow n = 15$	

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Example 8

Three numbers are consecutive terms of an arithmetic sequence. The sum of the three numbers is 45, and their product is 3240. Find the three numbers.

Let the three numbers be $u-d$, u, $u+d$	
$3u = 45$	Taking the sum of the numbers.
$\Rightarrow u = 15$	
$u(u^2 - d^2) = 3240$	Take the product.
$\Rightarrow 15^2 - d^2 = \frac{3240}{15} = 216$	Substitute $u = 15$ and divide by 15.
$\Rightarrow d^2 = 225 - 216 = 9$	
$\Rightarrow d = \pm 3$	The two values of d produce two possible sequences:
	12, 15, 18 or
The three numbers are 12, 15 and 18.	18, 15, 12

Example 9

The second term of an arithmetic sequence is 20 and the seventh term is 55. Find the first term and the common difference of the sequence.

$u_2 = u_1 + d = 20$	$u_n = u_1 + (n-1)d$
$u_7 = u_1 + 6d = 55$	
$\Rightarrow 5d = 35 \Rightarrow d = 7$	
$u_1 = 20 - 7 = 13$	Solving simultaneously.
Or	
$u_7 = u_2 + 5d \Rightarrow 5d = 55 - 20$	Write u_7 in terms of u_2.
$\Rightarrow 5d = 35 \Rightarrow d = 7$	Solve for d.
$u_1 = 20 - 7 = 13$	

The sum of an arithmetic sequence

Investigation 4

Miss Sandra, the grade 5 teacher, pairs up her students and gives each pair 55 cards numbered from 1 to 55. She tells the students that she wants them to use these cards to find the sum of the numbers $1 + 2 + 3 + \ldots + 55$.

Michela and Grisha start by laying out the cards in ascending order. Michela takes away the first card and the last card and notes that their sum is 56. Grisha then takes the first and last card from the cards that remain and notes that their sum is also 56. They continue to do this until just one card is left.

1. Which card will this be?
2. Using the information above, how would you determine the sum of the first 55 positive integers?
3. What if you wanted to find the sum of the first 1000 positive integers?
4. **Factual** Explain the importance of the actual number of terms added.
5. Repeat the process for finding the sum of:
   - **a** the first 100 even numbers
   - **b** the positive multiples of 3 less than 1000.
6. **Conceptual** How was Michela's and Grisha's method more efficient?

Reflect on Investigation 6 and explain how the method used is equivalent to the direct derivation for the sum of an arithmetic series containing n terms, with first term u_1 and common difference d as shown below.

$$\begin{array}{lllllllll} S_n = u_1 & + & u_1 + d & + & u_1 + 2d & + & \ldots & + & u_1 + (n-2)d & + & u_1 + (n-1)d \\ S_n = u_1 + (n-1)d & + & u_1 + (n-2)d & + & u_1 + (n-3)d & + & \ldots & + & u_1 + d & + & u_1 \\ \hline 2S_n = 2u_1 + (n-1)d & + & 2u_1 + (n-1)d & + & 2u_1 + (n-1)d & + & \ldots & + & 2u_1 + (n-1)d & + & 2u_1 + (n-1)d \end{array}$$

$$\Rightarrow 2S_n = n\left[2u_1 + (n-1)d\right]$$

$$\Rightarrow S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$$

This can be rewritten as follows:

$$S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$$

$$= \frac{n}{2}\left[u_1 + u_1 + (n-1)d\right]$$

$$= \frac{n}{2}\left[u_1 + u_n\right]$$

The sum of a finite arithmetic series is given by $S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right] = \frac{n}{2}\left[u_1 + u_n\right]$ where n is the number of terms in the series, u_1 is the first term, d is the common difference and u_n is the last term.

International-mindedness

Karl Friedrich Gauss (1777–1855) was a renowned German mathematician. It is said that when he was in primary school his teacher challenged him to find the sum of the numbers from 1 to 100. To the teacher's amazement, Gauss gave the correct answer almost immediately. He came to the answer by using the method used in investigation 4.

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How is intuition used in mathematics?

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Example 10

The first term of an arithmetic series is 5 and the last term is –51. The series has 15 terms. Find:

a the common difference

b the sum of the series.

a $-51 = 5 + 14d$ $d = \frac{-56}{14} = -4$	Using $u_n = u_1 + (n-1)d$.
b $S_{15} = \frac{15}{2}[5 + (-51)] = -345$	Using $S_n = \frac{n}{2}[u_1 + u_n]$.

Example 11

The first term of an arithmetic series is –7 and the fourth term is 23. The sum of the series is 689. Find the number of terms in the series.

$u_1 = -7$ $u_1 + 3d = 23 \Rightarrow d = \frac{23+7}{3} = 10$	Using $u_4 = u_1 + 3d$.
$S_n = 689 = \frac{n}{2}[-14 + (n-1) \times 10]$	
$\Rightarrow 10n^2 - 24n - 1378 = 0$ $\Rightarrow 5n^2 - 12n - 689 = 0$ $\Rightarrow (5n + 53)(n - 13) = 0$ $\Rightarrow n = 13$, since $n \in \mathbb{Z}^+$	Rearrange and solve for n.

Why can n not be a rational or a negative number?

Example 12

Find the value of $\sum_{r=1}^{28} 5r - 4$.

$u_1 = 1$ $u_{28} = 140 - 4 = 136$ $S_{28} = \frac{28}{2}(1 + 136) = 1918$	Substitute $r = 1$ and $r = 28$ to find the first and last terms. Using the formula $S_n = \frac{n}{2}[u_1 + u_n]$.

Example 13

The sum of an arithmetic series is given by $S_n = n(2n - 3)$. Find the common difference and the first three terms of the series.

$S_1 = u_1 = -1$ $S_2 = u_1 + (u_1 + d) \Rightarrow -2 + d = 2$ $d = 4$ $u_1 = -1$ $u_2 = 3$ $u_3 = 7$	Using $S_n = n(2n - 3)$. Using $S_2 = u_1 + u_2$.

Exercise 1B

1 Find the nth term of each of these sequences:

a 3, 8, 13, 18, …

b 101, 97, 93, 89, …

c $a - 3$, $a + 1$, $a + 5$, $a + 9$, …

d –20, –5, 10, 25, …

2 Find the terms indicated in each of these arithmetic sequences:

a 5, 11, 17, 23, … 15th term

b 10, 3, –4, –11, … 11th term

c a, $a + 2$, $a + 4$, $a + 6$, … 17th term

d 16, 12, 8, 4, … $(n + 1)$th term

3 Find the number of terms in each of these arithmetic sequences:

a 16, 11, 6, …, –64

b –108, –101, –94, …, 60

c –15, –19, –23, …, –95

d $2a + 5$, $2a + 3$, $2a + 1$, …, $2a - 23$

4 Determine the first term and the common difference of the arithmetic sequences that are generated by each of the following nth terms:

a $u_n = 5n - 7$

b $u_n = 3n + 11$

c $u_n = 6 - 11n$

d $u_n = 2a + 2n + 1$

5 The sixth term of an arithmetic sequence is 37 and the common difference is 7. Find the first term and the nth term.

6 The fifth term of an arithmetic sequence is 0 and the 15th term is 180. Find the common difference and the first term.

7 The sum of three consecutive terms of an arithmetic sequence is 24 and their product is –640. Find the three numbers.

8 Jung Ho earned €38,000 when he started his first job in the year 2000. He received a raise of €500 each consecutive year. Determine how much he earned in 2017? Evaluate in which year he would earn 50% more than his original salary?

9 Find the value of each of the following series:

a $3 - 3 - 9 - 15 - 21 - \ldots - 93$

b $31 + 40 + 49 + \ldots + 517$

c $(a - 1) + (a + 2) + (a + 5) + \ldots + (a + 146)$

10 Find the value of each of the following sums:

a $\sum_{r=1}^{50} 3r - 8$ **b** $\sum_{r=1}^{100} 7 - 8r$

c $\sum_{r=1}^{20} 2ar - 1$, where a is a constant

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11 Find the sums of the following sequences up to the term indicated:

a 4, –1, –6, … 15th term

b 3, 11, 19, … 10th term

c 1,–4, –9, … 20th term

12 Calculate the sum of an arithmetic series with 25 terms given that the fifth term is 19 and 10th term is 39.

13 The third term of an arithmetic sequence is –8, and the sum of the first 10 terms of the sequence is –230. Find:

a the first term of the sequence

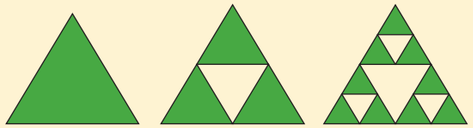
b the sum of the first 13 terms.

14 The sum of an arithmetic series is given by $S_n = 6n - 3n^2$. Find the common difference and the first four terms of the series.

15 Calculate the sum of all the odd numbers less than 300.

Investigation 5

The diagram below shows the first two iterations when constructing Sierpinski's triangle, named after the Polish mathematician Waclaw Sierpinski who first described it in 1915.



Stage 0 Stage 1 Stage 2

1 Construct the next iteration (Stage 3).

2 Copy and fill out the table below by following these instructions:

- Count the number of green triangles at each stage.
- If the sides of the triangle in stage 0 are each 1 unit long, what are the lengths of sides of green triangles at each of the following three stages? (Express your answers as rational numbers.)
- Now assume that the area of the triangle at Stage 0 is 1 unit². What is the area of each green triangle at each of the next three stages? (Leave answers in fractional form.)

Stage	0	1	2	3
Number of green triangles	1			
Length of one side of one green triangle	1			
Area of each green triangle	1			

3 **Factual** What patterns emerge from each of the three rows of the table?

4 **Factual** What do these three patterns have in common?

5 Based on your results, form a conjecture to obtain the numbers if you were to extend the table further to stages 4, 5, 6, etc.

6 **Conceptual** What has the mathematical model you have created enabled you to do?

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Is all knowledge concerned with identification and use of patterns?

Geometric sequences and series

In Investigation 5 you should have noticed that when filling out the table you would need to multiply the numbers in each row by a particular constant to obtain the following column. In other words, the ratio of a particular term to the previous term is a constant. Such sequences are known as geometric sequences.

If the ratio of two consecutive terms in a sequence is constant then it is a **geometric sequence** or a **geometric progression**. We call the constant ratio the **common ratio** and denote it by r.

HINT

The recursive equation for a geometric sequence is $u_n = u_{n-1} \times r$

Consider how a geometric sequence with first term u_1 and common ratio r grows:

First term u_1

Second term $u_2 = u_1 r$

Third term $u_3 = u_2 r = u_1 r^2$

Fourth term $u_4 = ur = u_1 r^3$

This leads to the general term $u_n = u_1 r^{n-1}$.

A geometric sequence with first term u_1 and common ratio r has **general term** $u_n = u_1 r^{n-1}$, $r \neq 1, 0, -1$, $u_1 \neq 0$.

Curiosities in geometric patterns

- What happens if you have a sequence with first term u_1 and common ratio 1?
- What if the common ratio is 0?
- And what happens if the common ratio is –1?

In the first case, the sequence is just made up of constant terms u_1. This is called a uniform sequence.

The next case is a sequence with first term u_1 and all the other terms are 0, which is a rather uninteresting sequence.

The third case leads to what is known as an oscillating sequence: $u_1, -u_1, u_1, -u_1, \ldots$

This oscillating sequence becomes particularly interesting if $u_1 = 1$, which then leads to the sequence $1, -1, 1, -1, 1, \ldots$

If we try to take the sum of this series we run into some curious and interesting results.

We want to look at the sum $S = 1 - 1 + 1 - 1 + 1 - 1 + \ldots$

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There are various ways of looking at this sum. Possibly the most intuitive way of finding this sum is by grouping the terms into pairs as follows:

$S = (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + \ldots = 0 + 0 + 0 + 0 + \ldots = 0$

But what happens if we pair the terms starting from the second term instead of the first?

$S = 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots$
$= 1 + 0 + 0 + 0 + 0 \ldots = 1$

Yet another result is obtained if we look at the series from a different perspective:

$S = 1 - (1 - 1 + 1 - 1 + 1 - 1 + \ldots)$
$= 1 - S$
$\Rightarrow 2S = 1$
$\Rightarrow S = \frac{1}{2}$

Why does this paradox arise and which is the correct answer? We have once more stumbled on the concept of infinity. If the number of terms were to be made finite, then the result would be 0 if there are an even number of terms, and 1 if the number of terms were odd, but an infinite sum never ends.

The next examples show how to use the general term formula for a geometric sequence to answer different types of questions.

International-mindedness

The series $S = 1 - 1 + 1 - 1 + 1 - 1 \ldots$ is known as Grandi's series, after the Italian mathematician Guido Grandi (1671–1742). You may want to look into the history and research on this sum by various mathematicians after its first appearance in Grandi's book published in 1703.

Example 14

Find the common ratio and write the next two terms of each sequence:

a 2.5, 5, 10, … **b** 9, 3, 1, … **c** $x, 2x^3, 4x^5, \ldots$

a $r = \frac{5}{2.5} = 2$

The next two terms are 20, 40.

b $r = \frac{3}{9} = \frac{1}{3}$

The next two terms are $\frac{1}{3}, \frac{1}{9}$.

c $r = \frac{2x^3}{x} = 2x^2$

The next two terms are $8x^7, 16x^9$.

Find r by calculating $\frac{u_2}{u_1}$.

Use the recursive equation to find the next two terms.

Example 15

Find the number of terms in each of these geometric sequences:

a 0.15, 0.45, 1.35, …, 12.15 **b** 440, 110, 27.5, …, 0.4296875

a $u_1 = 0.15, r = \frac{0.45}{0.15} = 3$

$u_n = 0.15 \times 3^{n-1} = 12.15$

$\Rightarrow 3^{n-1} = 81 = 3^4$

$\Rightarrow n = 5$

This sequence has 5 terms.

b $u_1 = 440, r = \frac{110}{440} = 0.25$

$u_n = 440 \times 0.25^{n-1} = 0.4296875$

$\Rightarrow n - 1 = 5$

$\Rightarrow n = 6$

This sequence has 6 terms.

Determine the value of r by computing $\frac{u_2}{u_1}$.

Use $u_n = u_1 r^{n-1}$ to find n.

Use technology to find the value of n.

x	$440 \times (0.25)^x$
1	110
2	27.5
3	6.875
4	1.71875
5	0.4296875
6	0.1074219
7	0.0268555

Example 16

The first term of a geometric sequence is 4 and the common ratio is –2. Determine which term has the value of –2048?

$4 \times (-2)^{n-1} = -2048$

$\Rightarrow n = 10$

Use technology to find the value of n.

x	$4 \times (-2)^{x-1}$
1	4
2	–8
3	16
4	–32
5	64
6	–128
7	256
8	–512
9	1024
10	–2048

HINT

This time the formula uses $(x - 1)$ in the exponent so the answer is $n = 10$

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Example 17

The fourth term of a geometric sequence is 54 and the sixth term is 486. Determine the possible values of the common ratio.

$u_4 = u_1 \times r^3 = 54$
$u_6 = u_1 \times r^5 = 486$
$\Rightarrow r^2 = \frac{486}{54} = 9$
$r = \pm 3$

Or

$u_6 = u_4 \times r^2$
$\Rightarrow r^2 = \frac{486}{54} = 9$
$\Rightarrow r = \pm 3$

Use $u_n = u_1 r^{n-1}$.

Divide the two expressions to obtain r^2.

Example 18

The first term of a geometric sequence is 16 and the common ratio is $\frac{1}{2}$.

Find the biggest term that is smaller than $\frac{1}{1000}$.

$16 \times \left(\frac{1}{2}\right)^{n-1} < 0.001$
$\Rightarrow n = 15$

$u_{15} = 16 \times \left(\frac{1}{2}\right)^{15}$

Alternatively, you can use your GDC.

Use technology to find the value of n.

x	$16 \times (0.5)^{(x-1)}$
1	16
2	8
3	4
4	2
.	.
.	.
13	0.00390625
14	0.001953125
15	0.000976563
16	0.000488281

The sum of a geometric sequence

When trying to find the value of the series $S = 1 + 3 + 9 + 27 + 81 + 243$, Max notices that this is a geometric series with common ratio 3, and that if he were to multiply the series by 3, he could more easily calculate the sum as follows:

$3S = 3 + 9 + 27 + 81 + 243 + 729$
$S = 1 + 3 + 9 + 27 + 81 + 243$
$\Rightarrow 3S - S = 2S = 729 - 1 = 728$
$S = 364$

Max then tried to generalize this result for a finite geometric series with common ratio r and having n terms as follows:

$S_n = u_1 + u_1 r + u_1 r^2 + u_1 r^3 + \ldots + u_1 r^{n-1}$
$rS_n = u_1 r + u_1 r^2 + u_1 r^3 + \ldots + u_1 r^{n-1} + u_1 r^n$
$\Rightarrow (1-r)S_n = u_1 - u_1 r^n$

$\Rightarrow S_n = \frac{u_1(1-r^n)}{1-r}, \quad r \neq 1$

HINT

This formula can also be written as follows:

$S_n = \frac{u_1(r^n - 1)}{r-1}, \quad r \neq 1.$

This makes calculations easier when $r > 1$.

The sum of a finite geometric series is given by

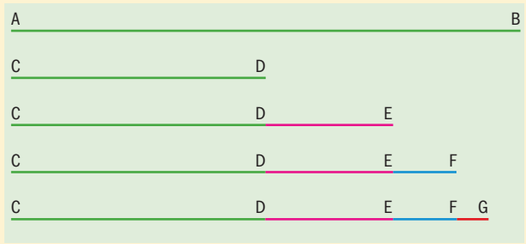
$S_n = \frac{u_1(1-r^n)}{1-r}, \quad r \neq 1$

where n is the number of terms, u_1 is the first term and r is the common ratio.

Investigation 6

In the diagram, AB represents a piece of string which is 100 cm long.

The string is cut in half and one of the halves, CD, is placed underneath. The remaining half is now cut in half and one of the halves, DE, is placed next to CD. The process is continued as shown in the diagram.



1 Copy and complete the table below.

Line segment	Length of string segment (cm)	Total length of segments (cm)
CD	50	50
DE	25	75
EF		
FG		

2 **Factual** As this process continues indefinitely, what do you notice about the length of the line segments? What about the total length of segments?

Modelling this scenario mathematically:

$CD = 50 \text{ cm}$

$DE = 50 \text{ cm} \times \frac{1}{2} = 25 \text{ cm}$

$EF = DE \times \frac{1}{2} = 50 \text{ cm} \times \left(\frac{1}{2}\right)^2 = 12.5 \text{ cm}$

$CD + DE + EF + FG = 50 + 50 \times \left(\frac{1}{2}\right) + 50 \times \left(\frac{1}{2}\right)^2 + 50 \times \left(\frac{1}{2}\right)^3$

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After four cuts have been made the sum of the length of string segments placed next to each other is a geometric sequence with four terms. Show that if n cuts are made this sum becomes $\frac{50\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\left(\frac{1}{2}\right)}$.

Enter this into a table as shown below to see what happens as n gets bigger.

n	$\frac{50\times(1-(0.5)^n)}{(1-0.5)}$
1	50
2	75
3	87.5
4	
5	
6	
7	
8	

3 What would happen if you repeated this experiment, but this time you cut CD to be $\frac{2}{3}$ of AB and DE to be $\frac{2}{3}$ of the remaining piece of string?

4 Repeat the process using CD to be $\frac{3}{4}$ of AB and DE to be $\frac{3}{4}$ of the remaining piece of string. What if the fraction used was $\frac{4}{5}$?

5 Write a short reflection on your results which includes answers to the following questions:

- **Factual** Why were you asked to change the length of the string cut?
- **Conceptual** How has this process helped you analyse the situation?
- **Conceptual** What conclusion could you draw from your analysis?

Convergent and divergent series

An infinite geometric series is **convergent** when the sum tends to a finite value as the number of terms gets bigger. If a geometric series does not converge it is said to be **divergent**.

In Investigation 6, the series always converged to 100 cm, the length of the original piece of string.

Investigation 7

In Investigation 6, you would have noticed that you had a geometric series in each case. You will now investigate a general geometric series in order to understand which conditions will make the series converge.

For a geometric series, we know that $S_n = \frac{u_1(1-r^n)}{1-r}$, $r \neq 1$.

Use technology to copy and complete the following table:

n	3^n	$(-2)^n$	$(1.5)^n$	$(0.5)^n$	$(-0.2)^n$	$(-0.75)^n$
1	3	−2	1.5	0.5	−0.2	−0.75
2						
3						
4						
5						
6						
7						
8						
9						
10						

1 Extend the table for different values of r^n and larger values of n.

2 **Conceptual** What role does the value of the common ratio play in a geometric series?

3 Use your results to justify the following statements:

a $r > 1 \Rightarrow r^n$ increases as n gets larger.

b $0 < r < 1 \Rightarrow r^n$ decreases as n gets larger.

c $r < -1 \Rightarrow r^n$ has a large absolute value, but its sign oscillates.

d $-1 < r < 0 \Rightarrow r^n$ has a small absolute value but its sign oscillates.

e When the value of r is close to (but still less than) 1, the value of r^n decreases more slowly but still gets close to zero when n gets larger.

f If $-1 < r < 1 \Rightarrow S_n \to \frac{u_1}{1-r}$ as the value of n gets larger.

The sum of n terms of a geometric series is $S_n = \frac{u_1(1-r^n)}{1-r}$, $r \neq 1$.

When $-1 < r < 1$, r^n approaches zero for very large values of n. The series therefore converges to a finite sum given by $S = \frac{u_1}{1-r}$.

The next examples demonstrate how to use the formulae for sums of finite and infinite geometric series.

Example 19

A geometric series has first term 3 and common ratio 2. Find the sum of the first five terms.

$S_5 = \frac{3(1-2^5)}{1-2} = \frac{3(1-32)}{-1} = 93$	Use the formula $S_n = \frac{u_1(1-r^n)}{1-r}$.

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Example 20

Calculate the geometric series given by $\sum_{i=1}^{7} 2\times\left(\frac{1}{2}\right)^i$.

$u_1 = 1,\ r = \frac{1}{2}$	Enter $i = 1$ and $i = 2$ to find u_1 and u_2 and hence r.
$S_7 = \frac{1\left(1-\left(\frac{1}{2}\right)^7\right)}{1-\frac{1}{2}} = \frac{1\left(1-\frac{1}{128}\right)}{\frac{1}{2}} \simeq 1.98$	Use the formula $S_n = \frac{u_1(1-r^n)}{1-r}$.
Or	The sum can be found using technology.
Using a GDC: $\sum_{i=1}^{7} 2\times\left(\frac{1}{2}\right)^i \simeq 1.98$	$\sum_{I=1}^{7}(2\times(1\div 2)^I) = 1.984375$

Example 21

Find two possible geometric sequences where the sum of the first two terms is 20 and the sum of the first four terms is 1640, and write the general term of each sequence.

$S_2 = u_1 + u_2 = u_1(1+r)$ $S_4 = \frac{u_1(1-r^4)}{1-r}$ $\Rightarrow \frac{S_4}{S_2} = \frac{(1-r^4)}{(1-r)(1+r)}$	$\frac{(1-r^4)}{(1-r)(1+r)} = \frac{(1-r^2)(1+r^2)}{(1-r^2)} = (1+r^2)$ Find the ratio of S_4 to S_2.
$\Rightarrow \frac{S_4}{S_2} = \frac{1640}{20} = 1+r^2$	Simplify and solve for r.
$\Rightarrow r^2 = 81$ $\Rightarrow r = \pm 9$	
$r = 9 \Rightarrow u_1 = \frac{20}{10} = 2$ or $r = -9 \Rightarrow u_1 = \frac{20}{-8} = -\frac{5}{2}$	Calculate u_1 for each value of r.
The two possible geometric sequences are given by the general terms: $u_n = 2\times 9^{n-1}$ or $u_n = \left(-\frac{5}{2}\right)\times 9^{n-1}$	

Example 22

The sum of the first n terms of a geometric sequence is given by $S_n = 7^n - 1$. Find the first term and the common ratio of the sequence.

$S_1 = 7 - 1 = 6 \Rightarrow u_1 = 6$ $S_2 = 49 - 1 = 48 \Rightarrow 6 + u_2 = 48 \Rightarrow u_2 = 42$ $r = \frac{42}{6} = 7$	Use formula given to find S_1 and S_2.

Example 23

Determine how many terms are required for the sum of the geometric series given by $\sum_{i=1}^{n} 3\times 2^i$ to exceed 1000.

$u_1 = 6,\ r = \frac{u_2}{u_1} = 2$	$S_n = \frac{u_1(1-r^n)}{1-r}$
$\frac{6(1-2^n)}{1-2} > 1000$ $\Rightarrow 2^n - 1 > \frac{1000}{6}$ $\Rightarrow 2^n > 167.\dot{6}$	Use technology to produce a table:
Using the table, $n = 8$.	
When 8 or more terms are added, the sum exceeds 1000.	

n	2^n
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256

Example 24

For what values of x does the series $\sum_{i=1}^{\infty}\left(1+\frac{x}{2}\right)^i$, $x \neq -2$ converge? Find the sum when $x = -1.5$.

$S = \left(1+\frac{x}{2}\right) + \left(1+\frac{x}{2}\right)^2 + \left(1+\frac{x}{2}\right)^3 + \ldots$	Write the first three terms of the series.
This is a geometric series where $u_1 = r = \left(1+\frac{x}{2}\right)$	Identify r.

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For convergence:	Use condition for convergence, ie $-1 < r < 1$.
$-1 < \left(1 + \frac{x}{2}\right) < 1$	
$\Rightarrow -2 < \frac{x}{2} < 0$	
$\Rightarrow -4 < x < 0$, but x cannot be -2	
When $x = -1.5$, the series converges to $S = \frac{-0.5}{1.5} = -\frac{1}{3}$.	Use the formula $S = \frac{u_1}{1-r}$.

Example 25

A geometric series converges to 8. The second term of the series is $-\frac{5}{2}$. Find the common ratio.

$S = \frac{u_1}{1-r} = 8$ $u_2 = u_1 r = -\frac{5}{2}$ $\Rightarrow 8(1-r) \times r = -\frac{5}{2}$	Use the formula $S = \frac{u_1}{1-r}$.
$16r^2 - 16r - 5 = 0$ $\Rightarrow (4r+1)(4r-5) = 0$ $\Rightarrow r = -\frac{1}{4}$ or $r = \frac{5}{4}$	Simplify and factorize.
But, since the series converges, $-1 < r < 1$ Hence the answer is $r = -\frac{1}{4}$.	

Exercise 1C

1 Write down the fifth term and the general term of each of the following sequences:

a 1, 3, 9, … **b** 8, 4, 2, …

c $\frac{x}{2}, \frac{x^3}{2}, \frac{x^5}{2}, \ldots$ **d** $-3, 3, -3, 3, \ldots$

2 Determine the common ratio and write the terms indicated in each of the following sequences:

a 63, 21, 7, … 6th term

b $243, \frac{81}{2}, \frac{27}{4}, \ldots$ 7th term

c $\frac{a}{2}, -\frac{a}{6}, \frac{a}{18}, \ldots$ 5th term

3 Determine the number of terms in each of the following sequences:

a 0.02, 0.06, 0.18, …, 3936.6

b $64, 32, 16, \ldots, \frac{1}{128}$

4 The fourth term of a geometric sequence is 6 and the seventh term is 48. Find the common ratio and the first term of the sequence.

5 The third term of a geometric sequence is 6 and the fifth term is 54. Find the two possible values of the common ratio and the sixth term of each sequence.

6 The first term of a geometric sequence is 9 and the fifth term is 16. Show that there are two possible sequences and find their common seventh term.

7 The numbers $3a + 1$, $a + 2$, and $a - 4$ are three consecutive terms of a geometric sequence. Find the two possible values of the common ratio.

8 The numbers $a - 1$, $a + 1$, and $a - 2$ are the fourth, fifth and sixth terms, respectively, of a geometric sequence. Find the common ratio and the first term of the sequence.

9 Find the following series for the number of terms stated:

a $3 - 1 + \frac{1}{3} - \frac{1}{9} + \ldots$ 6 terms

b $8 + 4 + 2 + 1 + \ldots$ 10 terms

c $0.1 + 0.03 + 0.009 + 0.0027 + \ldots$ 15 terms

d $0.1 - 0.03 + 0.009 - 0.0027 + \ldots$ 15 terms

10 Calculate:

a $\sum_{i=1}^{6} 7^{3-i}$ **b** $\sum_{i=0}^{n-1} 5 \times 10^i$

11 Show that a geometric sequence with first term 3 and seventh term $\frac{1}{243}$ has two possible sums to infinity and find them.

12 The sum of n terms of a certain series is given by $S_n = \left(-\frac{1}{2}\right)^n - 1$.

a Find the first three terms of the series.

b Show that the terms of the series are in geometric progression.

13 The second term of a geometric series is 28 and the third term is $28(1 - a)$.

Find the common ratio, given that the series converges and the sum of the first three terms is 147.

14 A length of material measures 2 m. It is cut into three lengths which are in geometric progression. The longest piece is twice as long as the shortest piece. Find the common ratio of the sequence and the exact length of the shortest piece.

15 Write the first four terms of the series $\sum_{i=0}^{\infty} (-1)^i \left(\frac{x}{2} + 1\right)^i$. Determine for what values of x this series converges. Find the value of the series when $x = -0.8$.

Developing inquiry skills

Go back to the original question about Koch's snowflake and try to address the following, assuming that the length of each side of the original triangle is 81 cm:

- Calculate the perimeter of the snowflake at each iteration.
- Calculate the area of the snowflake at each iteration.
- Tabulate the results and explain the number patterns that you observe.
- Create a model that helps you generalize the perimeter and area at any iteration.

TOK

How do mathematicians reconcile the fact that some conclusions conflict with intuition?

Although it might not be obvious, you have actually been exposed to arithmetic and geometric sequences and series in previous mathematics classes. Usually this was in the form of solving word problems. When we apply knowledge that was obtained by abstracting generalizations of mathematical concepts to real-life situations we

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are actually modelling the situation mathematically. The following investigations illustrate how arithmetic sequences can be hidden in everyday practices.

Investigation 8

Before the start of the school year, a stationer needs to stock up with notebooks. He has been given the following offers:

Provider A: Notebooks in packs of 20, at an offer of 6 for the price of 4, where each packet of 20 notebooks costs €10.

Provider B: Notebooks in packs of 100, at an offer of 3 for the price of 2, where each pack of 100 costs €48.

The stationer is considering stocking between 500 and 3000 notebooks, in multiples of 100.

1 **Factual** The stationer first looks at the offer made by Provider A and realizes that he would get the cheaper rate when the number of notebooks ordered are in a particular arithmetic sequence. Show that the stationer is correct.

2 **Factual** The stationer's wife tells him that the argument is true for the offer from Provider B. Is the wife also correct?

3 **Conceptual** They then compare costs incurred when buying notebooks from Provider A and from Provider B. They notice that for certain numbers of notebooks ordered it would be cheaper or the same rate if they were to order from Provider A. Determine which numbers they are referring to. How is this list of numbers different from the previous two answers?

4 **Conceptual** The stationer would like to divide his order of 1500 notebooks between the two providers. How can he divide up his order to minimize his cost price? If he sells the notebooks at 45 cents each, what would be his percentage profit? How does this compare to his percentage profit had he ordered all the notebooks from either of the providers?

5 How would this work help the stationer when making the order?

6 How realistic is the selling price fixed by the stationer?

Investigation 9

Students were given the following data and asked to work in groups to create a growth model for the shipment of smartphones worldwide from 2011 to 2016.

Year	Shipments worldwide in billions
2011	0.52
2012	0.74
2013	1.05

Students in group A decided that the data could be modelled using an arithmetic sequence by taking the average difference.

Students in group B decided that the data could be modelled using a geometric sequence by taking an average common ratio.

1 Create each growth model and determine the number of smartphones shipped in the years 2014 to 2016 as predicted by each model.

2 The actual shipments for the years 2014 to 2016 are given below. Which seems to be the better model for growth?

Year	Shipments worldwide in billions
2014	1.32
2015	1.46
2016	1.51

3 Use your models to predict shipment numbers up to and including the year 2025.

4 **Conceptual** How can you determine whether an arithmetic series grows faster than a geometric series? Which model looks more realistic and why?

The following examples illustrate how arithmetic and geometric sequences and series can be used to solve problems.

Example 26

The number of Facebook users at the end of 2008 was 145 million and growing at a rate of 3% per week. At the end of 2010 the number of Facebook users was 608 million.

a If the rate of growth had remained constant at 3% per week, determine the number of users at the end of 2010.

b The growth rate of 3% per week remained steady for 6 months and then dropped to 1.1% per week. This growth rate was maintained for another six months but then it dropped to 0.75% per week. Assuming that this rate was sustained for a whole year, show that this model better describes the recorded numbers.

c If the rate of growth dropped to 0.6% at the beginning of 2011 and remained steady, determine how long it would take for the number of users to reach 1 billion.

a The number of users after 2 years is $145 \times (1.03)^{104} \approx 3136$ million.

b The number of users after 6 months is $145 \times (1.03)^{26} \approx 313$ million.

The number of users at the end of 2009 is $312.706 \times (1.011)^{26} \approx 416$ million.

The number of users at the end of 2010 is $415.59 \times (1.0075)^{52} \approx 613$ million.

A growth of 3% weekly can be modelled using a geometric sequence where the value after 2 years (104 weeks) is given by $145 \times (1.03)^{104}$.

All answers are given to 3 significant figures.

Continued on next page

DRAFT

c The number of users at the end of 2010 is 613 million and the growth model is given by $613 \times (1.006)^n$.

We want $613 \times (1.006)^n = 1000$.

Using technology and table to solve for n gives:

The number of users will reach 1 billion during the year 2092, which is 82 years after the end of 2010.

Or use GDC to solve the equation.

Year	Number of years	Number of users 613×1.006^n
2011	1	616.678
2012	2	620.378068
2013	3	624.1003364
2014	4	627.8449384
2015	5	631.6120081
2016	6	635.4016801
.	.	.
.	.	.
.	.	.
2090	80	989.2337664
2091	81	995.169169
2092	82	1001.140184

Example 27

When Jacob turned 18 he had access to the money his grandparents had invested in a savings account. He decided to reinvest \$10 000 at a compound interest rate of 3% each year. He decided to add \$200 to this investment on his next birthday and each following birthday until he turned 25. Evaluate how much money was in his account just after his 25th birthday. Evaluate the total interest gained over this time.

On his 18th birthday Jacob had \$10 000.	
Just before his 19th birthday he had $10\,000(1.03)$	Amount in bank + interest for one year.
On his 19th birthday the amount was $10\,000(1.03) + 200$.	Value after 1 year + \$200.
Just before his 20th birthday: $(10\,000(1.03) + 200) \times 1.03$	
Just after his 20th birthday:	
$(10\,000(1.03) + 200) \times 1.03 + 200$	
$= 10\,000(1.03)^2 + 200(1.03)^1 + 200$	
After 7 years, just after his 25th birthday the amount would be:	Using pattern from years 1 and 2.
$10\,000(1.03)^7 + 200(1.03)^6 + 200(1.03)^5 + \ldots + 200$	
$= 10\,000(1.03)^7 + 200(\underbrace{(1.03)^6 + (1.03)^5 + \ldots + (1.03) + 1}_{\text{Geometric series}})$	Geometric series with $u_1 = 1$, $r = 1.03$ and $n = 7$.
$= 12\,298.73 + 200\left(\frac{1-(1.03)^7}{1-1.03}\right) = 12\,298.73 + 200(7.662..)$	
$= \$13\,831$ (to the nearest dollar)	
Money invested without interest $= 10\,000 + 200 \times 7 = 1140$	
Total interest $= 13\,831 - 1140 = \$2431$	

Exercise 1D

1 In 2010, a shop sold 220 televisions. Every six months the shop sold five more televi-sions, so that it sold 230 televisions in 2011, 240 in 2012, etc.

a Evaluate how many televisions the shop sold in 2017.

b Calculate the total number of televisions sold between 2010 and 2017 inclusive.

c The selling price of a television was €600 in 2010 but the selling price fell by €20 each year. In a particular year the number of televisions sold by the shop was half of the selling price of each television. Determine in which year this occurred.

2 Jane started working in 2000. On each successive year she received a salary increase equivalent to 1.5% of her previous salary. In 2011 her salary was €49,650. Determine her starting salary to the nearest €.

3 Carla traced her family tree back four generations. Carla's parents are the first generation back and her first set of ancestors. Carla's four grandparents are the second generation back and her second set of ancestors.

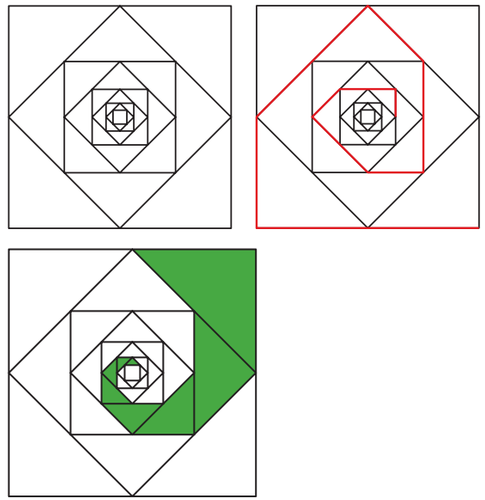
a How many ancestors are in Carla's family tree?

b Determine how many generations back she would have to trace to find more than 1 million ancestors.

4 Prisana is testing a recipe for a cake and tries it out several times, adjusting the amount of flour and sugar used each time. In the first recipe she uses 200 g of flour, and she decides to increase the weight of flour by 20 g in each trial. She has time to try out the recipe just 10 times. How many kg of flour would she need? She has 1.5 kg of sugar available and knows that the amount of sugar needed is usually half the amount of flour by weight. After the first two trials she decides to change the amount of sugar in each trial according to a geometric sequence. Evaluate the number of trials this would allow her to carry out. Explain how reliable her model is for using a geometric growth model for sugar content.

5 The first diagram shows a sequence of squares.

Starting with the largest square, the midpoints are joined to form the second square of the sequence. This process can be continued infinitely (in theory, but not in practice!)



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a If the sides of the largest square have a length of 2 units, calculate the side lengths of the second, third and fourth squares.

A spiral is formed by joining segments shown as red lines in the second diagram.

b Use your answers in part **a** to find the length of the spiral shown.

c Explain what happens to the length of the spiral if we continue the process infinitely.

A different spiral is formed by shading the triangles as shown in the third diagram.

d Find the total area of the shaded triangles.

e Determine the total area of the spiral formed if the process of forming squares and shading triangles is continued infinitely.

6 Gyuhun takes out a loan of $1500 to furnish his new student apartment.

The terms of the loan are that Gyuhun will pay equal monthly instalments. Interest is calculated monthly and is charged at 12% p.a. The loan is to be repaid in two years.

a Calculate the amount that Gyuhun has to pay each month if the first repayment is made one month after the money is borrowed and after interest is calculated.

b Evaluate how much, to the nearest dollar, Gyuhun actually has to pay for furnishing the apartment?

HINT

Per annum (p.a.) is a term often used in financial contexts and means "for each year".

7 An architect is designing a cinema that should hold 570 seats. The front row of the cinema can hold 30 seats and each consecutive row is to hold six seats more than the previous row.

a Evaluate how many rows the cinema can hold.

For optimal viewing, the floor of the cinema needs to be stepped so that the step between each row is 15 cm high and 95 cm deep. Assuming that the front row is 3 m away from the screen and the ceiling at the back row is 2.4 m high, determine:

b the horizontal distance from the screen to the top row

c the maximum height of the ceiling.

8 Ayla, Brynna and Cindy each receive €200 from their parents with the condition that they promise to invest it for at least 10 years. Ayla invests her money in an account with Rapid Bank that offers 5% simple interest annually. Brynna says it is better to invest in an account that offers compound interest because it grows quicker, so she uses an account with Quick Bank that offers 3.5% interest compounded annually. Cindy is not sure which offer is best so she invests €100 with Rapid Bank and €100 with Quick Bank. Answer the following questions giving your answers to the nearest €.

TOK

Do all societies view investment and interest in the same way?

What is your stance?

a Evaluate how much each investment is worth after 10 years.

b How much is each investment worth after 25 years?

c Determine after how many years the three investments yield approximately the same amount.

9 At the start of 2010 Karim had $5000 to invest. He decided to invest part of the money in a savings account that offered 1.5% simple interest per year. He added $1000 to this amount and fixed it for 10 years in bonds that offered 2.5% compound interest per year. The rest of the money he invested in shares.

a After one year the money invested in shares made a loss of 1%. Given that the total amount of money invested increased by $75, determine how much money Karim invested in each.

b At the end of the first year, Karim decided to sell the shares at their current value and reinvest the money in the savings account. Evaluate the total value of his investment at the end of 2020.

c Evaluate how much more money he would have made if he had divided up the $5000 equally between the savings account and bonds at the very start? (Give all your answers to the nearest dollar.)

10 A pharmaceutical company has developed a drug that fights a bacterial infection. The drug is to be administered four times per day, every six hours. It was found that six hours after administering, 37.5% of the original amount was still in the bloodstream. The maximum safe level of the drug in the bloodstream is 8 mg ml^{-1}.

a Construct a model to represent the amount of drug in the bloodstream at the end of day 1, ie immediately after the fourth administration.

b The company advises that the drug should not be administered for more than 10 days. Evaluate the maximum amount that should be administered to ensure that the amount of drug in the bloodstream does not exceed the safety level?

c The drug starts being effective when the amount of drug in the bloodstream is 7 mg ml^{-1}. Determine how many times the drug must be administered for this level to be reached.

Developing inquiry skills

You should now go back to the start of the chapter and answer the questions in the scenario that involves taking a loan for buying a new car.

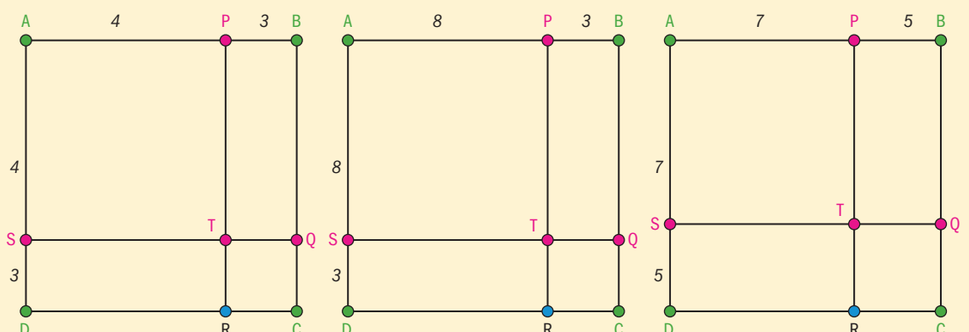
Developing your toolkit

Now do the Modelling and investigation activity on page XX.

1.3 Proof

Investigation 10a

Each of the three diagrams below, not drawn to scale, consists of a square ABCD of different sizes. Line PR is perpendicular to AB and DC and line SQ is perpendicular to AD and BC. Copy and complete the table on the next page.



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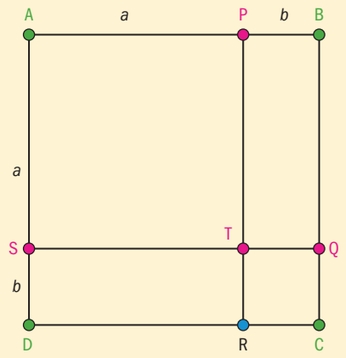
Area ABCD	Area APTS	Area BPQT	Area STRD	Area TQCR
$(3+4)^2=49$	4^2	3×4		

1 Describe the relationship between the areas of square ABCD, rectangles PBCT and TRDS, and square TQCR.

2 Rewrite the relationship above replacing words with numbers.

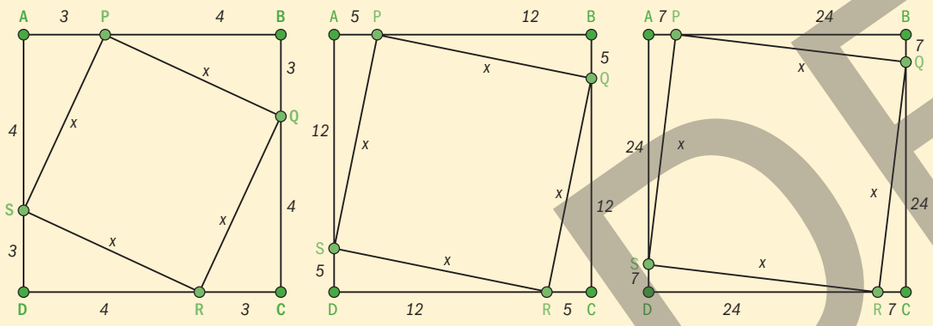
3 Now rewrite the relationship for the diagram on the right.

4 **Factual** What do you call this relationship? Why?



Investigation 10b

Each of the three diagrams below, not drawn to scale, consists of two squares ABCD and PQRS.
Use this information to copy and complete the table for each diagram.



Area ABCD	Area PQRS	Area $\triangle$PBQ	Area PQRS $+ 4\times$ area $\triangle$PBQ
$(3+4)^2=49$	x^2	6	
	x^2		
	x^2		

1 Describe the relationship between the areas of the squares and the triangles.

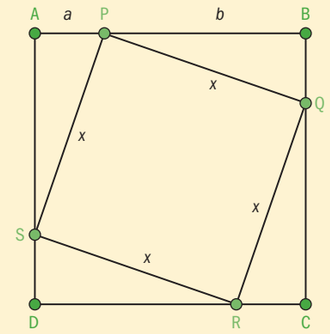
2 Rewrite the relationship above replacing words with numbers and making the area of PQRS the subject.

3 **Factual** What do you call each of these relationships? Why?

4 Determine the value of x in each case.

5 Now rewrite the relationship for the diagram on the right.

6 **Factual** What do you call this relationship? Why?



What is proof?

In Investigation 10a, you should have noticed that by making the question visual you were able to prove the identity $(a+b)^2 \equiv a^2+2ab+b^2$. By looking at a square which is divided into two smaller squares and two rectangles and comparing areas you came to a valid conclusion, and by then representing numbers with variables you could show that this identity is valid for all values of a and b. We can say that you have proved the statement $(a+b)^2 = a^2+2ab+b^2$ is true for all $a, b \in \mathbb{R}^+$ because the perpendiculars PR and SQ could be placed anywhere along the sides of square ABCD, which could also be as large or as small as you wanted.

Similarly, in Investigation 10b, you were able to validate the statement that in any right-angled triangle the lengths of the sides obey the relationship $a^2+b^2=c^2$, where c is the length of the hypotenuse of the triangle.

A **proof** in mathematics often consists of a logical set of steps that validates the truth of a general statement beyond any doubt.

There are many ways of presenting a proof and we will be looking at some of these ways in this section.

TOK

What is the role of the mathematical community in determining the validity of a mathematical proof?

Types of proof

Investigation 11

Copy and complete the table below and then suggest a conjecture.

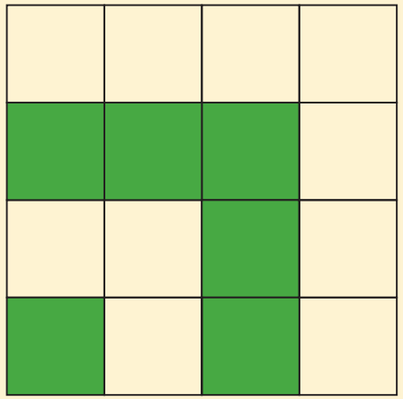
1	
$1+3$	
$1+3+5$	
$1+3+5+7$	

Continued on next page

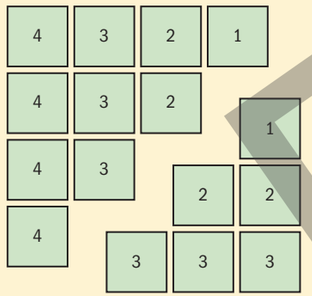
DRAFT

$1+3+5+7+9$	
$1+3+5+7+9+11$	

Now let's look at the same sequence visually.
Below you will find two visuals that represent a 4×4 square differently. In the first one, if you add the squares shaded white and green alternately you will obtain the sequence $1+3+5+7=16$.



The second visual is made up of the sequences $1+2+3+4$, and the sequence $1+2+3$, and when placed next to each other in the orientation shown they form a 4×4 square giving a sum of 16.



Combining the two visuals we can say that:

$$\begin{aligned} 4^2 &= 1+3+5+7 \\ &= (1+2+3+4)+(1+2+3) \\ &= \frac{4}{2}(1+4) + \frac{3}{2}(1+3) \\ &= 10 + 6 \end{aligned}$$

Seeing the question from this perspective should make your conjecture more valid. However, this alone is not a proof, because you cannot really generate a general square. A more rigorous proof of the conjecture is required and this is found in Example 28.

Example 28

Show that $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$.

$S = 1 + 3 + 5 + \ldots + (2n-3) + (2n-1)$ $S = (2n-1) + (2n-3) + (2n-5) + \ldots + 3 + 1$ $2S = \underbrace{2n + 2n + 2n + \ldots + 2n + 2n}_{n \text{ times}}$ $\Rightarrow 2S = 2n^2$ $\Rightarrow S = n^2$	Write out the sum in reverse order. Add the two sums and simplify.

A **direct proof** is a way of showing the truth of a given statement by constructing a series of reasoned connected established facts. In a direct proof the following steps are used:

- Identify the given statement.
- Use axioms, theorems, etc, to make deductions that prove the conclusion of your statement to be true.

The proof given in Example 28 consists of a set of reasoned steps that leads to the required result. Note that we did not just quote the sum of an arithmetic sequence with first term 1 and common difference 2, although by the definition of direct proof this would have been valid.

Example 29

Show that:

a The sum of an odd and even positive integer is always odd.
b The sum of two even numbers is always even.
c The sum of two odd numbers is always even.

a Let m and n be an odd and an even positive integer respectively. $\Rightarrow m = 2p - 1$ and $n = 2s$ where $p, s \in \mathbb{Z}^+$ $\Rightarrow m + n = 2p - 1 + 2s = 2(p + s) - 1$ This is an odd number since $p + s \in \mathbb{Z}^+$.	An odd positive integer can be written as $2k - 1$ where $k \in \mathbb{Z}^+$. An even positive integer can be written as $2k$ where $k \in \mathbb{Z}^+$.
b Let m and n be two even positive integers. $\Rightarrow m = 2p$ and $n = 2s$ where $p, s \in \mathbb{Z}^+$ $\Rightarrow m + n = 2(p + s)$ This is an even number since $p + s \in \mathbb{Z}^+$.	

Continued on next page

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c Let m and n be two odd positive integers. $\Rightarrow m = 2p - 1$ and $n = 2s - 1$ where $p, s \in \mathbb{Z}^+$ $\Rightarrow m + n = 2p + 2s - 2$ $\Rightarrow m + n = 2(p + s - 1)$ This is an even number since $p + s - 1 \in \mathbb{Z}^+$.	

Example 30

Show that $\left(x + \left(\frac{a}{2}\right)\right)^2 - \left(\frac{a}{2}\right)^2 \equiv x^2 + ax$.

LHS $\equiv \left(x^2 + 2\left(\frac{a}{2}\right)x + \frac{a^2}{4}\right) - \frac{a^2}{4}$ $\equiv x^2 + ax$	Expand and simplify.

In Example 30 we started from the left hand side and showed that this is equivalent to the right hand side. **When writing down a proof it is very important to work on one side of the statement only.** A proof is also valid if you work on each side consecutively to obtain the same result. This method is shown in the example below.

Example 31

Prove that $(n + 4)^2 - 3n - 4 = (n + 1)(n + 4) + 8$.

LHS $= n^2 + 8n + 16 - 3n - 4$ $= n^2 + 5n + 12$ RHS $= n^2 + n + 4n + 4 + 8$ $= n^2 + 5n + 12$ LHS = RHS Therefore $(n + 4)^2 - 3n - 4 = (n + 1)(n + 4) + 8$	Note that you work on each side separately and not in the same line.

Example 32

Prove that if the sum of the digits of a four-digit number is divisible by 3, then the four-digit number is also divisible by 3.

Let n be a four-digit number such that $n = a_3a_2a_1a_0$. We know that: $a_3 = p \times 10^3$ $a_2 = q \times 10^2$ $a_1 = r \times 10^1$ $a_0 = s \times 10^0$ $0 \leq p, q, r, s \leq 9$ and $p \neq 0$ and $p + q + r + s = 3k$, $k \in \mathbb{Z}$. $\Rightarrow n = p \times 10^3 + q \times 10^2 + r \times 10^1 + s \times 10^0$ and $s = 3k - p - q - r$ Therefore $n = p \times 10^3 + q \times 10^2 + r \times 10^1 + 3k - p - q - r$ $= (p \times 10^3 - p) + (q \times 10^2 - q) + (r \times 10^1 - r) + 3k$ $= p(10^3 - 1) + q(10^2 - 1) + r(10^1 - 1) + 3k$ $= 999p + 99q + 9r + 3k$ $= 3(333p + 33q + 3r + k)$ Since $(333p + 33q + 3r + k) \in \mathbb{Z}$ it follows that n is divisible by 3.	Since we are given that n is a four-digit number, $p \neq 0$. We are given that the sum of the digits is a multiple of 3.

Example 33

Show that $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \ldots = \frac{1}{3}$.

LHS $= \left(\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots\right) - \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots\right)$ $= \underbrace{\left(\frac{1}{2} + \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4}\right)^2 + \ldots\right)}_{\text{Converging geometric series}} - \underbrace{\left(\frac{1}{4} + \frac{1}{4}\left(\frac{1}{4}\right) + \frac{1}{4}\left(\frac{1}{4}\right)^2 + \ldots\right)}_{\text{Converging geometric series}}$ $= \frac{\frac{1}{2}}{1 - \frac{1}{4}} - \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$	Separate the given series into one series with positive terms and one with negative terms. Note that this is the difference of two converging geometric series. Use the formula for sum of converging geometric series and simplify.

DRAFT

Exercise 1E

1 Prove that $(a+b)^2+(a-b)^2=2(a^2+b^2)$.

2 Show that the product of two odd numbers is always an odd number.

3 Prove that a four-digit number is divisible by 9 if the sum of its digits is divisible by 9. Hence identify which of the numbers 3978, 5453, 7898, 9864, 5670 are divisible by 9 without carrying out any division.

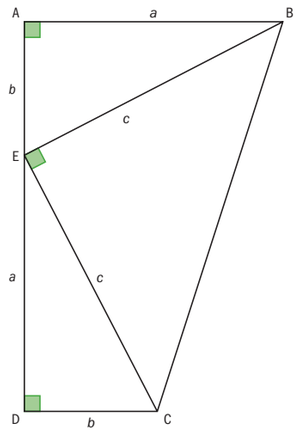
4 Show that $(a^2+b^2)(c^2+d^2)=(ad+bc)^2+(bd-ac)^2$.

5 Prove that $\frac{1}{3}-\frac{2}{9}+\frac{1}{27}-\frac{2}{81}+\frac{1}{243}-\frac{2}{729}+\ldots=\frac{1}{8}$.

6 Prove that the difference between the squares of two consecutive numbers is always an odd number.

7 Show that $\frac{1}{(n-1)}-\frac{1}{n}+\frac{1}{(n+1)}=\frac{n^2+1}{n(n^2-1)}$. Hence determine the value of $\frac{1}{5}-\frac{1}{6}+\frac{1}{7}$.

8 The diagram here shows a trapezium ABCD that has been divided into three triangles. Use your knowledge of areas and the diagram to show that $a^2+b^2=c^2$.



Proof by contradiction

Investigation 12

You will now look at the statement below and answer the questions:

If $n\in\mathbb{Z}$ and $5n+2$ is even, then n is even.

1 Write a direct proof of this statement.

2 Now assume that n is an odd number and rewrite the expression $5n+2$ to reflect this.

3 Simplify your expression and explain why this can never be an even number.

4 **Factual** How is the second method different to a direct proof?

In Investigation 12 you managed to find a different argument to prove the statement by using the **contrapositive.** You started by assuming that the second part of the statement is false and showed that this led to a contradiction, ie the first part of the statement was also false. In this case, the statement could easily be proved directly but this is not always the case, and sometimes you will need to use the second method to prove a statement correct.

HINT

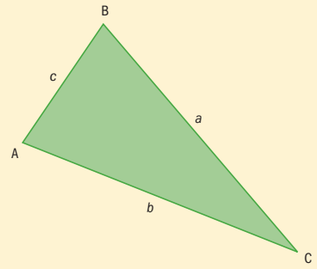
A proof by contradiction is built on the logical reasoning that If a proposition is true then its contrapositive is also true. That is, if in a statement a implies b then it is also true that if b is not true, this implies that a is not true.

Investigation 13

In geometry, the triangle inequality states that in any triangle ABC, the sum of the lengths of any two sides must be greater than or equal to the length of the remaining side.

Applying this to the triangle on the right we get the following:

$a+b\geq c$
$a+c\geq b$
$b+c\geq a$



1 Explain the situation when:

$a+b=c$
$a+c=b$
$b+c=a$

2 Look at triangle ADC in the quadrilateral on the right and write the triangle inequality in terms of AD and DC.

3 Now apply the triangle inequality to triangle ABC in terms of AC and AB.

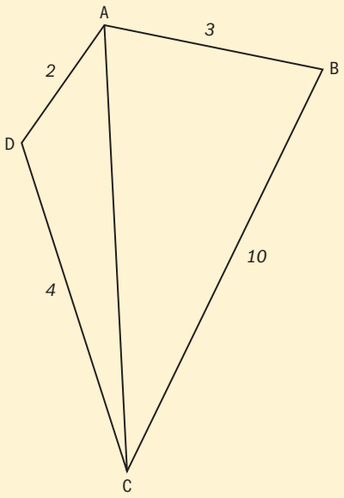
4 Look at the inequalities obtained in questions **2** and **3** and comment on your result.

5 Now draw diagonal BD and repeat steps **3** and **4** and **5** for triangles ABD and DBC.

6 What happens if you change the order of the sides?

7 **Factual** What do you conclude from this investigation?

8 **Conceptual** How else could you have come to the same conclusion?



When setting out a **proof by contradiction** you follow the following steps:

- Identify what is being implied by the statement.
- Assume that the implication is false.
- Use axioms, theorems, etc … to arrive at a contradiction.
- This proves that the original statement must be true.

In other words, the assumption contradicts either a given statement or something we already know to be true, or in some cases both.

TOK

What do mathematicians mean by mathematical proof, and how does it differ from good reasons in other areas of knowledge?

International-mindedness

Reductio ad absurdum (reduce to absurdity) is a term used to describe logical reasoning that attempts to disprove a statement by showing that it leads to an absurd result. In fact, this method can be traced back to the Greek philosopher Aristotle where he talks about "reduction to the impossible" in his book *Prior Analytics*.

DRAFT

The following examples will help you understand this method of proof so that you can then apply it in set tasks.

Example 34

Prove by contradiction:

a if the integer n is odd then n^2 is also odd

b if n^2 is even then n is also even

a Assume that n^2 is even. $\Rightarrow n^2 = 2k,\ k \in \mathbb{Z}$ $\Rightarrow n \times n = 2k$ But this cannot be true if n is odd as we know that the product of two odd numbers is also odd. Hence given that n is odd n^2 is also odd.	Start by making an assumption that the resulting statement is false.
b Assume that n is an odd integer. $\Rightarrow n = 2k \pm 1,\ k \in \mathbb{Z}$ $\Rightarrow n^2 = (2k \pm 1)^2 = 4k^2 \pm 4k + 1$ $\Rightarrow n^2 = 2(2k^2 \pm 2k) + 1 = 2p + 1$ Which is an odd number. Hence given that n^2 is an even integer n is also an even integer.	We want to prove that n^2 even $\Rightarrow n$ even. To use the contrapositive we start with n and deduce the contradiction ie that n^2 is not even.

Example 35

Show that $\sqrt{2}$ is irrational.

Assume that $\sqrt{2} = \frac{m}{n}$ where $m, n \in \mathbb{Z}$ and m, n have no common factors. $\Rightarrow 2 = \frac{m^2}{n^2}$ $\Rightarrow m^2 = 2n^2$ This means that m is an even integer. $\Rightarrow m = 2k$ and $m^2 = 4k^2$ This leads to: $m^2 = 4k^2 = 2n^2$ $\Rightarrow n^2 = 2k^2$ This means that n is an even integer.	Make the assumption that the statement is false, ie $\sqrt{2}$ is rational. Here we are using the fundamental theorem of arithmetic which states that every integer bigger than 1 is either prime or a multiple of primes and since m is an even integer it must contain a prime factor of 2.

International-mindedness

How did the Pythagoreans find out that $\sqrt{2}$ is irrational?

But it was assumed that m, n have no common factors and now we have found that they have a common factor of 2. Hence, we cannot find $m, n \in \mathbb{Z}$ that have no common factors to make $\sqrt{2} = \frac{m}{n}$ a rational number.	

Example 36

Prove that there is no $x \in \mathbb{R}$ such that $\frac{1}{x-2} = 1 - x$.

Assume there exists a real number a such that $\frac{1}{a-2} = 1 - a$. $\Rightarrow 1 = (a-2)(1-a)$ $\Rightarrow 1 = -a^2 + 3a - 2$ $\Rightarrow a^2 - 3a + 3 = 0$ $\Rightarrow a = \frac{3 \pm \sqrt{9-12}}{2} \notin \mathbb{R}$	Assume that a real solution $x = a$ exists. Solve for a. Apply the quadratic formula and conclude that a is not a real number since the number under the square root is negative. Show that a cannot be a real number.

Example 37

Prove that if $m, n \in \mathbb{Z}$, then $m^2 - 4n - 7 \neq 0$.

Assume that $m^2 - 4n - 7 = 0$. $\Rightarrow m^2 = 4n + 7$ $\Rightarrow m^2 = 4n + 6 + 1$ $\Rightarrow m^2 = 2(2n + 3) + 1$ This means that m^2 is an odd integer. But this means that m is an odd integer since an even integer squared is even. Let $m = 2p + 1$ where $p \in \mathbb{Z}$. Then $(2p + 1)^2 - 4n - 7 = 0$ $\Rightarrow 4p^2 + 4p + 1 - 4n - 7 = 0$ $\Rightarrow 2p^2 + 2p - 2n - 3 = 0$ $\Rightarrow 2(p^2 + p - n) = 3$ Since $p, n \in \mathbb{Z}$ this cannot be true as 3 is not even. Hence $m^2 - 4n - 7 \neq 0$ given that $m, n \in \mathbb{Z}$.	Assume that the statement is false. Rearrange the equation. Since $n \in \mathbb{Z} \Rightarrow 2n + 3 \in \mathbb{Z}$.

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Sometimes we encounter statements that seem to be true, and for every example that we consider the statement seems to hold. Although such examples are good to verify a statement or conjecture, they are not sufficient to *prove* the statement. It takes only one example that contradicts the statement to justify that the statement is wrong.

A **counterexample**, or **counterclaim**, is an acceptable "proof" of the fact that a given statement is false.

Some examples follow to demonstrate how counterexamples can be used.

Example 38

Show by a counterexample that the following statements are not true.

a If $n \in \mathbb{Z}$ and n^2 is divisible by 4, then n is divisible by 4.

b If $n \in \mathbb{Z}$ then $n^2 + 1$ is a prime number.

c If an integer is a multiple of 10 and 12 then it is a multiple of 120.

a When $n = 6$, $n^2 = 36$ which is divisible by 4, but 6 is not divisible by 4. **b** $n = 3 \Rightarrow n^2 + 1 = 10$ which is not a prime number. **c** 60 is a multiple of both 10 and 12 but it is not a multiple of 120.	Again, the statement is true for many integers but we only need one counterexample.

Exercise 1F

In questions **1** to **9**, prove the statements by contradiction.

1 For all $n \in \mathbb{Z}$, if n^2 is odd then n is also odd.

2 $\sqrt{3}$ is irrational.

3 $\sqrt[5]{2}$ is irrational.

4 For all $p, q \in \mathbb{Z}$, $p^2 - 8q - 11 \neq 0$.

5 For all $a, b \in \mathbb{Z}$, $12a - 6b \neq 0$.

6 If $a, b, c \in \mathbb{Z}$, where c is an odd number and $a^2 + b^2 = c^2$, then either a or b is an even number.

7 If $n, k \in \mathbb{Z}$, then $n^2 + 2 \neq 4k$.

8 If r is an irrational number and q is a rational number then $p + q$ is also irrational.

9 Given that m and n are positive integers, it follows that $m^2 - n^2 \neq 1$.

10 Show by a counterexample that the following statements are not true:

a $(m + n)^2 \neq m^2 + n^2$

b If a positive integer is divisible by a prime number then the number is not prime.

c $2^n - 1$ is a prime number for all $n \in \mathbb{N}$.

d $2^n - 1$ is a prime number for all $n \in \mathbb{Z}^+$.

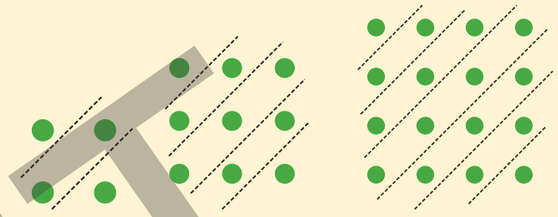
e The sum of three consecutive positive integers is always divisible by 4.

f The sum of four consecutive positive integers is always divisible by 4.

Proof by induction

Investigation 14

Look at the diagrams and answer the questions.



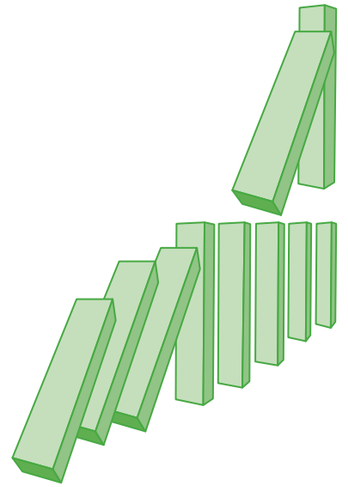
1 Each of the three diagrams represents a series. Write them down.

2 If the diagrams were to continue, what would the next three diagrams be?

3 Write a conjecture based on your findings.

4 Prove your conjecture using a direct proof.

In Investigation 14, you used a visual representation of a series to make a conjecture about a special series and then prove it. In this case, you were able to prove the conjecture directly, but there are times when such a direct proof is not possible. Sometimes we need to revert to a different proof which is called **proof by induction.**



To illustrate the principle of proof by induction, imagine two dominoes placed standing at a distance less than half their length, as illustrated in the diagram on the right. If the first domino is knocked over it will fall and cause the second domino to fall with it. This is the starting point of the process and is called the **basic step**.

Now assume that the domino in the kth place falls if the domino before it (in the $(k - 1)$th position) falls. This assumption is the second step in the process.

If we were to add another domino at the end of the k dominoes, this last domino will also fall. This analogy represents the final step of the process which is called the **inductive step**.

We can then finalize our argument by stating that at the start of the process it was shown that the first domino caused the second domino to fall. We can now use the second and third step that a third domino placed behind the first step will also fall, and again using the two steps, a fourth domino will also topple over. We can continue repeating this process as many times as we want. In other words, we have shown that we can have as many dominoes as we like and they will all fall if the first domino knocks over the second domino.

We apply the dominoes analogy to mathematics to prove the statement in Investigation 14. It should be noted that the visual could have started from a previous step, ie with just one green circle.

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In Example 39, we are going to use this as the basic step, so that we have the proof for all positive integer values of n.

Example 39

Prove by mathematical induction that $1 + 2 + 3 + \ldots + (n-1) + n + (n-1) + \ldots + 3 + 2 + 1 = n^2$.

$P(n)$: $1 + 2 + 3 + \ldots + (n-1) + n + (n-1) + \ldots + 3 + 2 + 1 = n^2$, $n \in \mathbb{Z}^+$	We start by making a statement $P(n)$ which we need to prove true for certain values of n (usually positive integers).
When $n = 1$	This is the basic step.
LHS $=1$	
RHS $=1$	
Since LHS = RHS $\Rightarrow P(1)$ is true	This is where we make the assumption. Note the wording just before we substitute for n in the statement.
Assume that $P(n)$ is true for some value of k, $k \geq 1$, $k \in \mathbb{Z}^+$ ie	
$1 + 2 + 3 + \ldots + (k-1) + k + (k-1) + \ldots + 3 + 2 + 1 = k^2$	
When $n = k + 1$ LHS	In the inductive step we *must* use the assumption to show that the statement is also true for $k+1$.
$= 1 + 2 + 3 + \ldots + (k-1) + k + (k+1) + k + (k-1) + \ldots + 3 + 2 + 1$	
$= 1 + 2 + 3 + \ldots + (k-1) + k + (k-1) + \ldots + 3 + 2 + 1 + (k+1) + k$	
$= k^2 + (k+1) + k$	
$= k^2 + 2k + 1$	
$= (k+1)^2$	
Since $P(1)$ was shown to be true and it was also shown that if the statement is true for some $n = k$, $k \in \mathbb{Z}^+$, it is also true for $n = k + 1$, it follows by the principle of mathematical induction that the statement is true for all positive integers.	This final statement completes the proof and should always be included.

Why is the basic step important?

The principle of mathematical induction is very rigorous, provided that all the steps are used. If the basic step is left out, we can end up with erroneous results. Suppose you were asked whether 10^n is a multiple of 7 and you try using mathematical induction without the basic step. Here is what you would obtain:

Assume $10^k = 7a$ for some n, $a \in \mathbb{Z}^+$.

We then move to the inductive step $10^{k+1} = 10 \times 10^k$ and using the assumption would give $10^{k+1} = 10 \times 10^k = 10 \times 7a = 7(10a)$. Since a is a positive integer, $10a$ is a positive integer also, so 10^n is a multiple of 7 for all positive integers. Of course, we know that this is not true because $10 = 2 \times 5 \Rightarrow 10^n = 2^n \times 5^n$ and since 2 and 5 are prime numbers 7 will never divide 10^n exactly.

Incorrect use of the inductive step

Proof by induction is also not valid if the assumption is not used in the inductive step, as shown below to prove that $11^n - 6$ is a multiple of 5.

$P(n)$: $11^n - 6 = 5a$, where n, $a \in \mathbb{Z}^+$.

Basic step:

When $n = 1$, LHS $= 11 - 6 = 5$.

Therefore, the statement is true for $n = 1$.

Assumption:

Assume that the statement is true for some $k \in \mathbb{Z}^+$, $k \geq 1$.

ie $11^k - 6 = 5b$, where n, $b \in \mathbb{Z}^+$.

Inductive step:

When $n = k + 1$:

LHS

$= 11^{k+1} - 6$

$= 11 \times 11^k - 6$

$= (5 + 6) \times 11^k - 6$ Write 11 as $5 + 6$

$= 5 \times 11^k + 6 \times 11^k - 6$ Distribute 11^k

$= 5 \times 11^k + 6(11^k - 1)$ 6 is a common factor

$= 5 \times 11^k + 6(11 - 1)(11^{k-1} + 11^{k-2} + 11^{k-3} + \ldots + k + 1)$

$= 5 \times 11^k + 60(11^{k-1} + 11^{k-2} + 11^{k-3} + \ldots + k + 1)$

$= 5(11^k + 12(11^{k-1} + 11^{k-2} + 11^{k-3} + \ldots + k + 1))$

$= 5m$

This is the result required and the mathematics above is correct. However, the assumption was not used to obtain the result. **Hence the proof by mathematical induction is incorrect.** The correct solution is shown in Example 40.

HINT

In the chapter review at the end of this chapter you will be guided to prove that $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1})$

Example 40

Use mathematical induction to prove that $11^n - 6$ is a multiple of 5.

$P(n)$: $11^n - 6 = 5a$, where n, $a \in \mathbb{Z}^+$	Write the statement that you want to prove.
When $n = 1$	Basic step, show statement is true for $n = 1$.

Continued on next page

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LHS = $11 - 6 = 5$	
Therefore, the statement is true for $n = 1$.	
Assume that the statement is true for some $k \in \mathbb{Z}^+$, $k \geq 1$.	Make assumption and then make 11^k the subject of the formula.
ie $11^k - 6 = 5b$, where $n, b \in \mathbb{Z}^+$	
$\Rightarrow 11^k = 5b + 6$	
When $n = k + 1$:	Inductive step.
LHS	
$= 11^{k+1} - 6$	Let $n = k + 1$.
$= 11 \times 11^k - 6$	Rewrite the LHS in terms of 11^k.
$= 11(5b + 6) - 6$	Use the assumption.
$= 55b + 66 - 6$	Simplify to obtain the required result, in this case write it as a product with a factor of 5.
$= 55b - 60$	
$= 5(11b - 12)$	
Since $P(1)$ was shown to be true and it was also shown that if the statement is true for some $n = k$, $k \in \mathbb{Z}^+$, it is also true for $n = k + 1$, it follows by the principle of mathematical induction that the statement is true for all positive integers.	Write final statement.

Example 41

Prove the following statements using mathematical induction.

a The sum of the first n terms of an arithmetic sequence with first term u_1 and common difference d is given by $S_n = \frac{n}{2}(2u_1 + (n-1)d)$.

b The sum of the first n terms of a geometric sequence with first term u_1 and common ratio r is given by $S_n = \frac{u_1(1-r^n)}{1-r}$.

a

$P(n)$: $S_n = \frac{n}{2}(2u_1 + (n-1)d)$	
LHS: When $n = 1$, $S_1 = u_1$	Basic step.
RHS: When $n = 1$, $\frac{1}{2}(2u_1 + (1-1)d) = u_1$	
LHS = RHS	
Therefore $P(1)$ is true.	Assumption.
Assume that $S_k = \frac{k}{2}(2u_1 + (k-1)d)$ for some $k \in \mathbb{Z}^+$.	Substitute k in the statement.
When $n = k + 1$	Inductive step.
$S_{k+1} = S_k + u_{k+1}$	Use assumption to obtain result
$= \frac{k}{2}(2u_1 + (k-1)d) + u_1 + kd$	
$= ku_1 + \frac{k}{2}(k-1)d + u_1 + kd$	Use algebra to simplify.
$= u_1(k+1) + kd\left(\frac{k-1}{2} + 1\right)$	
$= u_1(k+1) + kd\left(\frac{k+1}{2}\right)$	
$= \left(\frac{k+1}{2}\right)(2u_1 + kd)$	
$= \left(\frac{k+1}{2}\right)(2u_1 + ((k+1)-1)d)$	
Since $P(1)$ was shown to be true and it was also shown that if the statement is true for some $n = k$, $k \in \mathbb{Z}^+$, it is also true for $n = k + 1$, it follows by the principle of mathematical induction that the statement is true for all positive integers.	Write the final statement.

b

$P(n)$: $S_n = \frac{u_1(1-r^n)}{1-r}$	
LHS: When $n = 1$, $S_1 = u_1$	Basic step.
RHS: When $n = 1$, $\frac{u_1(1-r)}{1-r} = u_1$	
LHS = RHS	
Therefore $P(1)$ is true.	Assumption.
Assume that for some $k \in \mathbb{Z}^+$ $S_k = \frac{u_1(1-r^k)}{1-r}$	Substitute k in the statement.
When $n = k + 1$	Inductive step.
$S_{k+1} = S_k + u_{k+1}$	Use assumption to obtain result.
$= \frac{u_1(1-r^k)}{1-r} + u_1 r^k$	
$= \frac{u_1(1-r^k) + u_1 r^k(1-r)}{1-r}$	$S_{k+1} = \frac{u_1(1-r^{k+1})}{1-r}$.
$= \frac{u_1(1-r^k + r^k - r^{k+1})}{1-r}$	Use algebra to simplify.
$= \frac{u_1(1-r^{k+1})}{1-r}$	
Since $P(1)$ was shown to be true and it was also shown that if the statement is true for some $n = k$, $k \in \mathbb{Z}^+$, it is also true for $n = k + 1$, it follows by the principle of mathematical induction that the statement is true for all positive integers.	Write the final statement.

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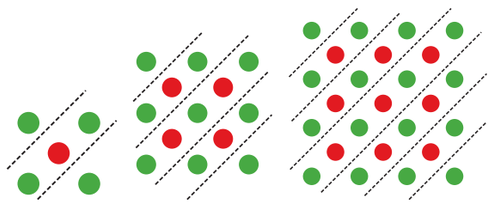
Example 42

Use mathematical induction to prove that $3^{2n} + 7$ is divisible by 8 for all $n \in \mathbb{N}$.

$P(n)$: $3^{2n} + 7 = 8A$	Note that since the statement holds for $n \in \mathbb{N}$ we have to start with 0.
When $n = 0$,	
LHS $= 3^0 + 7 = 8$	
So $P(0)$ is true.	
Assume that the statement is true for some $k \in \mathbb{N}$, $k \geq 0$.	Assumption.
ie $3^{2k} + 7 = 8A$	
$\Rightarrow 3^{2k} = 8A - 7,\ A \in \mathbb{Z}^+$	
When $n = k + 1$:	Use the assumption.
LHS $= 3^{2(k+1)} + 7$	
$= 9 \times 3^{2k} + 7$	
$= 9(8A - 7) + 7$	
$= 72A - 63 + 7$	
$= 72A - 56$	
$= 8(9A - 7)$	$(9A - 7) \in \mathbb{Z}^+$, since $A \in \mathbb{Z}^+$.
$= 8B$	
Since $P(0)$ was shown to be true and it was also shown that if the statement is true for some $n = k$, $k \in \mathbb{Z}^+$, it is also true for $n = k + 1$, it follows by the principle of mathematical induction that the statement is true for all positive integers.	

Exercise 1G

1 Use the diagrams to answer the questions below.



a Each of the three diagrams represents a series.

i Write down a series based on the line divisions.

ii Write down a series based on colour.

b If the diagrams were to continue what would the next two series be?

c Write a conjecture based on your findings.

d Prove your conjecture using a direct proof.

e Prove your conjecture using the principle of mathematical induction.

2 Use mathematical induction to prove the following statements:

a $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{3}n(n+1)(n+\frac{1}{2})$

b $1 - 4 + 9 - 16 + \ldots + (-1)^{n+1}n^2 = (-1)^{n+1}\frac{n(n+1)}{2}$

c $\sum_{i=0}^{n} 2^i = 2^n - 1$

d $9^n - 1$ is divisible by 8 for all $n \in \mathbb{N}$.

e $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$

f $n^3 - n$ is divisible by 3 for all $n \in \mathbb{N}$.

g $\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

h $n^3 - n$ is a multiple of 6 or all $n \in \mathbb{Z}^+$.

i $2^{n+2} + 3^{2n+1},\ n \in \mathbb{Z}^+$ is divisible by 7.

j $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

k $\sum_{r=1}^{n} r(r+1) = \frac{n}{3}(n+1)(n+2)$

l $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$

3 Given the statements below, decide how best you can prove or disprove the statement. Some statements can be proved in more than one way; you should attempt both ways and then decide which is the more elegant proof, and why.

a Prove that $(4n + 3)^2 - (4n - 3)^2$ is divisible by 12 for all positive integers n.

b $n^2 + 37n + 37$ is a prime number.

c $1^3 + 3^3 + 5^3 + \ldots + (2n - 1)^3 = n^2(2n^2 - 1)$

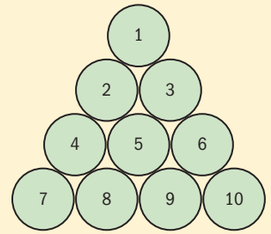
d $1\times2 + 2\times3 + 3\times4 + \ldots + (n-1)\times n = \frac{n(n^2-1)}{3}$

e $n^3 - n$ is divisible by 3 for all values of $n \in \mathbb{Z}^+$.

1.4 Counting principles and the binomial theorem

Investigation 15

Mary creates a fun game for practising some mathematics. She arranges 10 cups, numbers them as shown in the diagram on the right, and places one marble just outside cup number 1. She then writes the following instructions.



Instructions for play:

- The number of marbles you place in each cup is equal to the number of the cup multiplied by the number of marbles in the previous cup.
- The starting point is cup 1, where you will multiply the number on the cup by the number of marbles outside of cup 1.

1 If you were to follow these instructions, find how many marbles would be placed in:

a cup number 2 **b** cup number 3

c cup number 5 **d** cup number 8.

2 If Mary places another two rows underneath this arrangement, how many cups would there be in total?

3 How can you represent the number of marbles that would be placed in the last cup?

4 Comment on your results.

5 How would you represent the number of marbles in the last cup if there were n cups in total?

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Factorial notation

The numbers in Investigation 15 got large very quickly. We can denote these numbers mathematically as follows:

The first marble outside the cup = $u_0 = 1$.

For the other cups, $u_n = n \times u_{n-1}$.

If you were to build up each term of this sequence, you would end up with the result $u_n = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$.

Some mathematical problems about arrangements and combinations involve counting techniques that use this sequence. A simpler way to denote the sequence is to use factorial notation where $u_n = n!$, $u_0 = 0! = 1$.

Here are the first five factorial numbers:

$0! = 1$

$1! = 1 \times 0! = 1$

$2! = 2 \times 1! = 2$

$3! = 3 \times 2! = 6$

$4! = 4 \times 3! = 24$

$$n! = n \times (n-1)! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

This pattern lends itself to calculating expressions with very large numbers, as shown in the following examples.

Example 43

Find the value of these expressions:

a $\frac{7!}{5!}$ **b** $\frac{3!}{5!}$ **c** $\frac{8! \times 4!}{10!}$ **d** $\frac{7! \times 5!}{10! \times 6!}$

a $\frac{7!}{5!} = \frac{7 \times 6 \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = 42$

b $\frac{3!}{5!} = \frac{\cancel{3} \times \cancel{2} \times \cancel{1}}{5 \times 4 \times \cancel{3} \times \cancel{2} \times \cancel{1}} = \frac{1}{20}$

c $\frac{8! \times 4!}{10!} = \frac{4!}{10 \times 9} = \frac{4 \times 3 \times 2 \times 1}{10 \times 9} = \frac{4}{15}$

d $\frac{7! \times 5!}{10! \times 6!} = \frac{1}{10 \times 9 \times 8 \times 6} = \frac{1}{4320}$

You can also find these using technology as follows:

```
7!÷5!                    42
3!÷5!                  0.05
```

```
(8!×4!)÷10!              4/15
(7!×5!)÷(10!×6!)       1/4320
```

Example 44

Simplify the following.

a $\frac{n(n+1)!}{n!}$ **b** $\frac{n! - (n-1)!}{(n+1)!}$

a $\frac{n(n+1)!}{n!} = \frac{n(n+1) \times n!}{n!} = n(n+1)$

Rewrite $(n+1)!$ as $(n+1) \times n!$

b $\frac{n! - (n-1)!}{(n+1)!} = \frac{n(n-1)! - (n-1)!}{(n+1) \times n \times (n-1)!}$

$= \frac{\cancel{(n-1)!}(n-1)}{(n+1) \times n \times \cancel{(n-1)!}} = \frac{(n-1)}{n(n-1)}$

Rewrite $n!$ as $n(n-1)!$ and $(n+1)!$ as $(n+1) \times n \times (n-1)!$

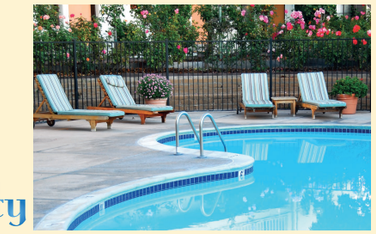
Permutations and combinations

Investigation 16a

Angela is creating invitation cards for her birthday party. She has three images which she wants to use on her invitation. She puts the images in a row as shown below.

16

Pool Party



She would like to consider all the possible ways of arranging these images in one line on a card and then choose her favourite.

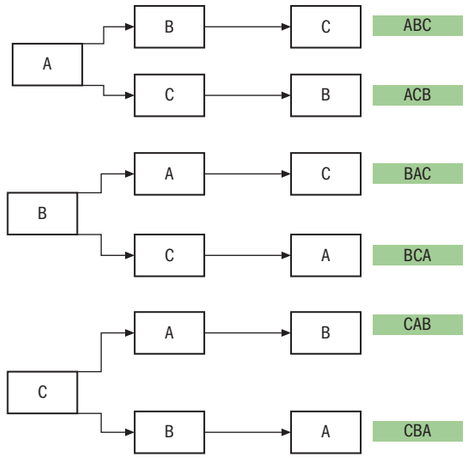
1. How many arrangements can she choose from?
2. How did you come up with your answer?
3. Angela realizes that she should also include the address, and she still wants to leave the four objects in a line. In how many ways can this be done?
4. She then decides that she does not need to include her address on invitations to her cousins, and realizes that if she makes all invitations individual she will have just enough different invitation cards for all her guests. How many people is she going to invite to her party?
5. What happens if she wants to invite another friend to the party? Explain your answer.

DRAFT

Most probably, when responding to Investigation 16a, all the different arrangements (called **permutations**) were listed. Here is another way of reasoning out the responses:

Suppose you want to find the total number of arrangements of the letters A, B and C. You have three letters to choose from for the first letter.

Having chosen this first letter, you have two choices for the second letter, and then you are left with only one letter to complete the whole set. In other words, Angela has $3 \times 2 \times 1$ ways of designing the invitation cards for her cousins. You can think of this method as filling boxes as shown here, starting from left to right.



This reasoning can be extended to deduce that the number of ways in which n distinct objects can be arranged in a row is

$n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1 = n!$

The number of ways of arranging n distinct objects in a row is $n!$

Now suppose that you have five different letters and you want to find the number of possible arrangements of just three of these letters. You can choose the first letter in five ways, the second letter in four ways and the third letter in three ways, giving:

$5 \times 4 \times 3 = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \frac{5!}{(5-3)!}$ ways.

Using the same reasoning, you can deduce that the number of ways of arranging three objects chosen out of n distinct objects would be $n \times (n-1) \times (n-2) = \frac{n!}{(n-3)!}$. And generalizing even further:

The number of permutations of r objects out of n distinct objects is given by ${}^nP_r = \frac{n!}{(n-r)!}$.

Investigation 16b

Let's revisit Investigation 16a. Angela now wants to choose a photo of the pool for her third image. She has five pool photos and decides to choose two different images, one for the invitation to relatives and another one for the invitation to friends.

1 In how many ways can she choose the first photo?

2 In how many ways can she choose the second photo?

3 It is irrelevant which photo to use on the two sets of invitations. In how many ways can she choose two photos out of five?

4 Why is this answer different to arranging two photos chosen out of five?

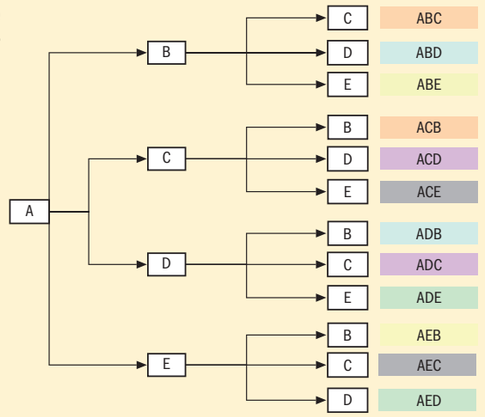
TOK

How many different tickets are possible in a lottery?

What does this tell us about the ethics of selling lottery tickets to those who do not understand the implications of these large numbers?

Investigation 16c

Let's consider what happens if we want to choose three letters out of five and represent these on a chart similar to the one discussed for permutations. If the first letter chosen is A then we have the chart shown here.



1 What do you notice about the colour coded arrangements on the right?

2 What would you expect to notice if the first letter chosen had been B?

3 What if you were to consider all the possible permutations?

4 If the order of choosing the letters is not important, how can you derive the number of combinations?

When the order of arrangements is not relevant we speak about combinations.

The number of ways of choosing (when order is not important) r objects from n distinct objects is ${}^nC_r = \frac{n!}{r!(n-r)!}$

HINT

In some books you may find the notation $\binom{n}{r}$ for ${}^nC_r$.

In this book we will use the latter which is the notation you will encounter in IB exams.

Example 45

a In how many ways can the letters of the word *candle* be arranged?

b In how many ways can a group of four boys and three girls be arranged?

c In how many ways can a group of four boys and three girls be arranged if no girls are to be next to each other?

a There are six letters in the word *candle*, which can be arranged in 6! = 720 ways.

b Seven children can be arranged in 7! = 5040 ways.

c 4! ways of arranging the boys.

3! ways of arranging the girls.

If the girls are to be separated, then each pair of girls must be separated by at least one boy.

There are five positions each girl can fill, at each end or the slots between boys, as shown in the diagram, giving the number of combinations ${}^5C_3$.

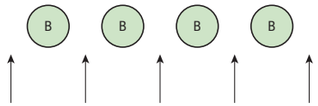
Using a calculator:

```
6!
                720
7!
               5040
```

Continued on next page

DRAFT

The total number of arrangements is therefore $4! \times 3! \times {}^5C_3 = 1440$.



So total number of arrangements is $4! \times 3! \times 10 = 1440$

Example 46

How many four-digit numbers can be made using each of the following digits only once?

a 5, 6, 7 and 8 **b** 5, 6, 7, 8 and 9 **c** 7, 8, 9 and 0

a ${}^4P_4 = 4! = 24$

b ${}^5P_4 = \dfrac{5!}{(5-4)!} = 120$

c $3 \times {}^3P_3 = 3 \times 3! = 18$

There are 4! ways of creating a four-digit number with these digits using each digit only once.

The number cannot start with 0.

So, there are three ways of choosing the first digit, and the next three digits can be chosen from the three remaining digits which include 0.

Example 47

There are eight boys and five girls who attend the Senior Mathematics Club. Find how many ways the teacher can choose a team of six students to represent the school in a competition if:

a There are no gender restrictions.

b The team is to be made up of three girls and three boys.

c At least two of each gender are included in the team.

a ${}^{13}C_6 = \dfrac{13!}{6!(13-6)!} = 1716$

b ${}^8C_3 \times {}^5C_3 = 560$

c We cannot have a team with only one girl or with no girls.

We also cannot have a team with only one boy.

We have to use combinations as the order of choosing is not important.

We now need to choose three boys out of eight and three girls out of five.

From the total number of ways of choosing the team (answer **a**) we need to exclude these three combinations, ie

${}^{13}C_6 - {}^8C_5\,{}^5C_1 - {}^8C_6\,{}^5C_0 - {}^8C_1\,{}^5C_5 = 1400$

Or

There are three ways to consider, two boys and four girls or three boys and three girls or four boys and two girls, giving

${}^8C_2 \times {}^5C_4 + {}^8C_3 \times {}^5C_3 + {}^8C_2 \times {}^5C_4 = 1400$

We cannot have a team with no boys as there are not enough girls to form a team.

Using technology:

```
13C6
                    1716
8C3×5C3
                     560
1716−8C5×5C1−8C6×5C0▷
                    1400
```

Exercise 1H

1 Copy and complete the table below, simplifying expressions as shown in the first row.

$8! - 6!$	$6!(56-1) = 39\,600$
$9! + 8!$	
$7! - 6!$	
$6! + 5!$	
$(n+1)! - n!$	
$n! - (n-1)!$	
$n! + (n-1)!$	
$(n+1)! + n!$	

2 Find the value of:

a $\dfrac{8!}{4 \times 6!}$ **b** $\dfrac{4! \times 5!}{3! \times 6!}$ **c** $\dfrac{10! \times 8!}{11! \times 6!}$

3 Simplify the following:

a $\dfrac{(n+1)!}{n!-(n+1)!}$ **b** $\dfrac{n!+(n+1)!}{n!}$ **c** $\dfrac{(n!)^2-1}{n!-1}$

4 Show that $\dfrac{(2n+2)!(n!)^2}{\left[(n+1)!\right]^2(2n)!} \equiv \dfrac{2(2n+1)}{n+1}$.

5 Solve for $n \in \mathbb{Z}^+$, ${}^nC_2 = 66$.

6 Solve the equation $16(n-1)! = 5n! + (n+1)!$ where $n \in \mathbb{Z}^+$.

7 On a bookshelf there are four mathematics books, three science books, two geography books and four history books. The books are all different.

a In how many different ways can the books be arranged on the shelf?

b In how many ways can the books be arranged so that books of the same subject are grouped together?

8 A safe has two dials, one with 26 letters and one with the digits 0 to 9.

In order to open the safe, Rose has to choose a code consisting of three distinct letters followed by two distinct digits. Determine how many different safe codes are possible.

9 A delegation of five students is to be selected for a Model United Nations conference. There are 10 boys and 13 girls to choose from.

a In how many different ways can a delegation be chosen if there are no restrictions?

b If the team is to include at least one girl and one boy, in how many ways can the delegation be selected?

10 **a** How many four-digit numbers can be made using the digits 0, 1, 3, 4, 5, 8 and 9?

b How many of the four-digit numbers have no repeated digits?

c How many four-digit even numbers can be made using the digits?

d How many of these even numbers are divisible by 5?

DRAFT

11 Graeme is training for a 10 km run. He has six different routes to choose for his training and he trains four times a week. He calculates that he will just manage to run a different set of routes each week leading up to his next race.

How many weeks are there before Graeme's race?

12 A group of 12 people want to go to a concert. They can travel in a small car that takes one driver and one passenger and two cars each taking one driver and four passengers. If there are five drivers in the group, in how many different ways can they travel?

The binomial theorem

Investigation 17

Copy and complete this table by using repeated algebraic multiplication.

$(1+x)^i$	Constant	Coefficient of x	Coefficient of x^2	Coefficient of x^3	Coefficient of x^4	Coefficient of x^5
$(1+x)^0$	1	-	-	-	-	-
$(1+x)^1$	1	1	-	-	-	-
$(1+x)^2$						
$(1+x)^3$						
$(1+x)^4$						
$(1+x)^5$						

1 Comment on any patterns that you recognize.

2 Rearrange the numbers so that rather than forming a right-angled triangle, they form an isosceles triangle with 1 at the top vertex.

3 What new patterns do you notice now?

4 If you extended this pattern, what would you get in the next row?

5 Verify your response to question **4** using algebraic multiplication.

Investigation 18

Consider the expansion $(1+x)^3 = (1+x)(1+x)(1+x)$.

1 How is the constant term obtained in this expansion?

2 How is the term in x obtained?

3 What about the term in x^2?

4 And the term in x^3?

5 Summarize your responses using mathematical notation.

6 **Conceptual** Repeat the process for the expansion $(a+x)^3 = (a+x)(a+x)(a+x)$

7 Write a conjecture for obtaining the expansion of

$$(a+x)^n = \underbrace{(a+x)(a+x)(a+x)\ldots(a+x)}_{n \text{ factors}}$$

The binomial theorem states that for all $n \in \mathbb{Z}^+, a, x \in \mathbb{R}$.

$$(a+x)^n = {}^nC_0a^n + {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 + \ldots + {}^nC_ra^{n-r}x^r + \ldots + {}^nC_nx^n$$

$$= \sum_{r=0}^{n} {}^nC_ra^{n-r}x^r$$

Investigation 19

1 Write the expansion of $(1+x)^n$ using combinations.

2 Find how many terms this expansion has when:

a $n = 4, 6, 10$ **b** n is even

c $n = 3, 5, 7$ **d** n is odd.

3 The pattern found in Investigation 17 is known as Pascal's triangle. The following properties of this pattern were found:

- There is a line of symmetry going down the middle of the numbers.
- Each row starts and ends with 1 and each of the other numbers is the sum of the two numbers above it to either side.

The expansion of $(1+x)^5$ can be written as follows:

$$(1+x)^5 = {}^5C_0x^0 + {}^5C_1x + {}^5C_2x^2 + {}^5C_3x^3 + {}^5C_4x^4 + {}^5C_5x^5$$

a Use this expansion and the one for $(1+x)^6$ to verify the two properties above.

b Write the expansions for $(1+x)^n$ and $(1+x)^{n+1}$.

c Use the same method as in part **a** to show that the two properties above hold.

d **Conceptual** How can you explain the patterns in Pascal's triangle by considering the general expansion of the binomial expansion?

TOK

Why do we call this Pascal's triangle when it was in use before Pascal was born?

Are mathematical theories merely the collective opinions of different mathematicians, or do such theories give us genuine knowledge of the real world?

Example 48

Find the values of a, b and c in the following identities:

a $(1-2x)^5 \equiv 1 + ax + bx^2 + cx^3 + \ldots - 32x^5$

b $\left(1+\frac{x}{3}\right)^a \equiv 1 + bx + cx^2 + \ldots + \left(\frac{x}{3}\right)^8$

c $(2+ax)^7 \equiv b + 224x + cx^2 + 70x^3 + \ldots$

a $(1-2x)^5 \equiv {}^5C_0 + {}^5C_1(-2x) + {}^5C_2(-2x)^2 + {}^5C_3(-2x)^3 + \ldots + {}^5C_5(-2x)^5$

$= 1 + 5\times(-2x) + 10\times(-2x)^2 + 10\times(-2x)^3 + \ldots + 1\times(-2x)^5$

$= 1 - 10x + 40x^2 - 80x^3 + \ldots - 32x^5$

$\Rightarrow a = -10,\ b = 40$ and $c = -80$

You should know the first five rows of Pascal's triangle.

$(1-2x)^5 = (1+(-2x))^5$

Continued on next page

DRAFT

b $\left(1+\frac{x}{3}\right)^a \equiv 1+bx+cx^2+\ldots.+\left(\frac{x}{3}\right)^8$

$\Rightarrow a=8$

$\left(1+\frac{x}{3}\right)^8 \equiv 1+{}^8C_1\left(\frac{x}{3}\right)+{}^8C_2\left(\frac{x}{3}\right)^2+\ldots+\left(\frac{x}{3}\right)^8$

$=1+\frac{8!}{1!7!}\times\frac{x}{3}+\frac{8!}{2!6!}\times\frac{x^2}{9}+\ldots+\left(\frac{x}{3}\right)^8$

$=1+\frac{8}{3}x+\frac{28}{9}x^2+\ldots+\left(\frac{x}{3}\right)^8$

$\Rightarrow b=\frac{8}{3}$ and $c=\frac{28}{9}$

c $(2+ax)^7 \equiv 2^7+{}^7C_1\times 2^6\times(ax)+{}^7C_2\times 2^5\times(ax)^2+{}^7C_3\times 2^4\times(ax)^3+\ldots$

$=128+\frac{7!}{1!6!}\times 2^6ax+\frac{7!}{2!5!}\times 2^5a^2x^2+\frac{7!}{3!4!}\times 2^4a^3x^3+\ldots$

$=128+448ax+672a^2x^2+560a^3x^3+\ldots$

$\Rightarrow 448a=224$

$\Rightarrow a=\frac{1}{2}$, $b=128$ and $c=168$

Example 49

Find the coefficient of x^3y^3 in the expansion of $(x+3y)^6$.

The general term in the expansion of $(x+3y)^6$ is given by ${}^6C_rx^{6-r}(3y)^r$.

We want ${}^6C_rx^{6-r}(3y)^r=Ax^3y^3 \Rightarrow r=3$.

Then $A={}^6C_3\times 3^3=\frac{6!}{3!3!}\times 27=540$.

Example 50

Use the binomial theorem to expand $(2x+3y)^5$. Hence find the value of 2.03^5 correct to 5 decimal places.

$(2x+3y)^5=(2x)^5+5(2x)^4(3y)+10(2x)^3(3y)^2+10(2x)^2(3y)^3+5(2x)(3y)^4+(3y)^5$

$=32x^5+240x^4y+720x^3y^2+1080x^2y^3+810xy^4+243y^5$

When $x=1$, $y=0.01$ we obtain

$(2.03)^5=32+2.40+0.0720+0.001\,080+0.000\,008\,10+0.000\,000\,0243$

$\simeq 32.47309$ (to 5 decimal places)

Example 51

Find the term independent of x in the expansion of $\left(x^2-\frac{1}{2x}\right)^6$.

The general term in the expansion of $\left(x^2-\frac{1}{2x}\right)^6$ is given by:	Write the general term of the expansion.
${}^6C_r\left(x^2\right)^{6-r}\left(-\frac{1}{2x}\right)^r={}^6C_r\left(-\frac{1}{2}\right)^r x^{12-2r-r}$	
For the term independent of x:	For the term independent of x, the total power of x must be 0.
$12-3r=0 \Rightarrow r=4$	
${}^6C_4\left(x^2\right)^{6-4}\left(-\frac{1}{2x}\right)^4={}^6C_4\left(-\frac{1}{2}\right)^4x^0=\frac{6!}{4!2!}\times\frac{1}{16}=\frac{15}{16}$	Give your answer as an exact fraction.

Exercise 1I

1 Write the first four terms in the binomial expansion of:

a $\left(1-\frac{x}{3}\right)^{11}$ **b** $\left(1+\frac{x}{2}\right)^7$ **c** $\left(x+\frac{2}{x}\right)^8$

2 In each of the following binomial expressions, write down the required term.

a fifth term of $(a-2b)^{10}$

b third term of $\left(a+\frac{4}{a^2}\right)^{11}$

c fourth term of $\left(x-\frac{2y}{x}\right)^8$

3 Find the term independent of x in the expansion of $\left(x-\frac{2}{x^2}\right)^{12}$.

4 Use the binomial theorem to expand $\left(2-\frac{x}{5}\right)^4$. Hence find the value of $(1.99)^4$ correct to 5 decimal places.

5 Find the term in x^6 in the expansion of $\left(x^2-\frac{1}{x}\right)^6$.

6 **a** Expand $\left(x+\frac{y}{x}\right)^5$.

b Find the coefficient of x^3y^2 in the expansion of $(2x+y)\left(x+\frac{y}{x}\right)^5$.

7 Write in factorial notation:

a the coefficient of x^4 in the expansion of $(1+x)^{n+1}$

b the coefficient of x^3 in the expansion of $(1+2x)^n$.

c Find n, given that these two coefficients are equal.

8 **a** Express $\left(\sqrt{3}-\sqrt{2}\right)^5$ in the form of $a\sqrt{3}+b\sqrt{2}$ where $a, b \in \mathbb{Z}$.

b Express $\left(\sqrt{2}-\frac{1}{\sqrt{5}}\right)^4$ in the form $a+b\sqrt{10}$, $a, b \in \mathbb{Q}$.

c Express $\left(1+\sqrt{5}\right)^7-\left(1-\sqrt{5}\right)^7$ in the form $a\sqrt{5}$, $a \in \mathbb{Z}$.

9 Find the value of the following by choosing an appropriate value for x in the expansion of $(1+x)^n$.

a ${}^nC_0-2\times{}^nC_1+4\times{}^nC_2-8\times{}^nC_3+\ldots+(-1)^r2^r\times{}^nC_r+\ldots+(-1)^n2^n\times{}^nC_n$

b ${}^nC_0+{}^nC_1+{}^nC_2+{}^nC_3+\ldots{}^nC_r+\ldots+{}^nC_n$

DRAFT

Generalization of the binomial expansion

It was around 1665 that Isaac Newton generalized the binomial theorem to allow for negative and fractional exponents. Let's try to examine this using some facts which were established earlier in this chapter.

Consider the geometric series $1 + x + x^2 + x^3 + \ldots$, where x is not equal to 0.

For which values of x does this series converge?

What is the sum to infinity for this series when it converges?

We can write the answers to these two questions as follows:

For $-1 < x < 1$, $S = \dfrac{1}{1-x}$

In other words:

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

If we want to expand $(1-x)^{-2}$ we could say that this is equivalent to $((1-x)^{-1})^2 = (1-x)^{-1}(1-x)^{-1}$

$= (1 + x + x^2 + x^3 + \ldots)(1 + x + x^2 + x^3 + \ldots)$

$= 1 + x + x^2 + x^3 + x^4 + \ldots$ multiplying the terms in the left bracket by 1

$\quad + x + x^2 + x^3 + x^4 + \ldots$ multiplying the terms in the left bracket by x

$\quad + x^2 + x^3 + x^4 + \ldots$ multiplying the terms in the left bracket by x^2

$\ldots$

$= 1 + 2x + 3x^2 + 4x^3 + \ldots$

$\Rightarrow (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

Similarly, we can repeat the process to obtain the expansion of $(1-x)^{-3}$:

$(1-x)^{-3} = (1-x)^{-2}(1-x)$

$= (1 + 2x + 3x^2 + 4x^3 + \ldots)(1 + x + x^2 + x^3 + \ldots)$

$= 1 + 2x + 3x^2 + 4x^3 + \ldots$

$\quad + x + 2x^2 + 3x^3 + \ldots$

$\quad + x^2 + 2x^3 + \ldots$

$\ldots$

$= 1 + 3x + 6x^2 + 10x^3 + \ldots$

Newton generalized this result for negative and rational exponents of the binomial theorem as follows:

The binomial expansion for $(1-x)^{-n}$ for $n \in \mathbb{Z}^+$ and $-1 < x < 1$ is given by the infinite series:

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots$$

TOK

Is it possible to know things about which we can have no experience, such as infinity?

International-mindedness

We cannot simply take it for granted that we can multiply out two infinite series easily. Gustav Dirichlet, a German mathematician, proved how this can be done.

The binomial expansion for $(1+x)^{\alpha}$ for $\alpha = \dfrac{p}{q} \in \mathbb{Q}$ and $-1 < x < 1$ is given by the infinite series:

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots + \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-r+1)}{r!}x^r + \ldots$$

Investigation 20

1 **Factual** Show that the generalized form for negative integer powers given above can be written as

$$(1-x)^{-n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r x^r$$

The table below includes the results obtained above.

$(1-x)^i$	Constant	Coefficient of x	Coefficient of x^2	Coefficient of x^3	Coefficient of x^4	Coefficient of x^5
$(1-x)^{-1}$	1	1	1	1	1	1 …
$(1-x)^{-2}$	1	1	3	4	5	6…
$(1-x)^{-3}$	1	−3	+6	−10…		
$(1-x)^{-4}$						
$(1-x)^{-5}$						

2 Use Newton's generalization to verify that the coefficients shown in the table are correct.

3 Apply Newton's generalization to copy and complete the table.

4 **Factual** Show that the generalized form of the binomial theorem for fractional powers can be written as

$$(1+x)^{\alpha} = \sum_{r=0}^{\infty} {}^{\alpha}C_r x^r$$

5 Use the generalization to find the expansion of $\sqrt{(1+x)}$ and $\sqrt{(1-x)}$.

Example 52

Expand the following up to the term in x^3.

a $\sqrt{(1+2x)}$, for $|x| < \dfrac{1}{2}$ **b** $\dfrac{2}{(1-3x)}$, for $|x| < \dfrac{1}{3}$

a $\sqrt{(1+2x)} = (1+2x)^{\frac{1}{2}}$

$= 1 + \frac{1}{2}(2x) + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{(2x)^2}{2!} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{(2x)^3}{3!} + \ldots$

$= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3$

Using $(1+x)^{\alpha} = 1 + \alpha x + \dfrac{\alpha(\alpha-1)}{2!}x^2 + \dfrac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots$

Simplify.

Continued on next page

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b $\dfrac{2}{(1-3x)} = 2(1-3x)^{-1}$

$= 2\left(1+(1)(3x)+\dfrac{(1)(2)}{2!}(3x)^2+\dfrac{(1)(2)(3)}{3!}(3x)^3+\ldots\right)$

$= 2\left(1+3x+9x^2+27x^3+\ldots\right)$

$= 2+6x+18x^2+54x^3+\ldots$

Using

$(1-x)^{-n} = 1+nx+\dfrac{n(n+1)}{2!}x^2 + \dfrac{n(n+1)(n+2)}{3!}x^3+\ldots$

$n \in \mathbb{Z}^+$ and $|x| < \dfrac{1}{3}$

Example 53

Use the binomial expansion to show that $\sqrt{\dfrac{1+x}{1-x}} \simeq 1+x+\dfrac{1}{2}x^2$, $|x|<1$.

$\sqrt{\dfrac{1+x}{1-x}} = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$

$(1+x)^{\frac{1}{2}} = 1+\left(\dfrac{1}{2}\right)x+\left(\dfrac{1}{2}\right)\left(-\dfrac{1}{2}\right)\dfrac{x^2}{2!}+\ldots$

$= 1+\dfrac{x}{2}-\dfrac{x^2}{8}+\ldots$

Using

$(1+x)^{\alpha} = 1+\alpha x+\dfrac{\alpha(\alpha-1)}{2!}x^2 + \dfrac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\ldots$

where $\alpha = \dfrac{1}{2}$

We only need the expansion until the term in x^2.

Using

$(1+x)^{\alpha} = 1+\alpha x+\dfrac{\alpha(\alpha-1)}{2!}x^2$

$(1-x)^{-\frac{1}{2}} = 1+\left(-\dfrac{1}{2}\right)(-x)+\left(-\dfrac{1}{2}\right)\left(-\dfrac{3}{2}\right)\dfrac{(-x)^2}{2!}+\ldots$

$= 1+\dfrac{x}{2}+\dfrac{3x^2}{8}+\ldots$

where $\alpha = -\dfrac{1}{2}$, and noting that we have $-x$ inside the brackets.

We only need the expansion until the term in x^2.

$(1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}} \simeq \left(1+\dfrac{x}{2}-\dfrac{x^2}{8}+\ldots\right)\left(1+\dfrac{x}{2}+\dfrac{3x^2}{8}+\ldots\right)$

Multiply the two expansions up to and including terms in x^2.

$(1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}} \simeq \begin{cases} 1+\dfrac{x}{2}+\dfrac{3x^2}{8} \\ +\dfrac{x}{2}+\dfrac{x^2}{4} \\ -\dfrac{x^2}{8} \end{cases} \ldots \simeq 1+x+\dfrac{x^2}{2}$

TOK

"Mathematics may be defined as the economy of counting. There is no problem in the whole of mathematics which cannot be solved by direct counting."

-E. Mach

To what extent do you agree with this quote?

Exercise 1J

1 Expand the following up to the term in x^3, given that $|x| < \dfrac{1}{2}$.

a $\dfrac{1}{(1+x)}$ **b** $\dfrac{1}{(1-2x)^2}$

c $\dfrac{2}{(1+2x)}$ **d** $\dfrac{2}{(1-x)^3}$

2 Find the first four terms of each of the following expansions where $|x| < \dfrac{1}{10}$:

a $\sqrt{(1+2x)}$ **b** $(1+x)^{\frac{3}{2}}$

c $(1-3x)^{-\frac{1}{2}}$ **d** $2(1+x)^{\frac{1}{3}}$

3 Show that $\sqrt{\dfrac{1-x}{1+x}} \simeq 1-x+\dfrac{x^2}{2}-\dfrac{x^3}{2}$, where $|x|<1$.

4 Show that $\dfrac{x}{(1+x)^2} \simeq x-2x^2+3x^3-4x^4$.

5 Find the binomial expansion of $(2-3x)^{-3}$, $|x| < \dfrac{2}{3}$.

6 a Find the first four terms of the binomial expansion of $\sqrt{1-4x}$, $|x| < \dfrac{1}{4}$.

b Show that the exact value of $\sqrt{1-4x}$ when $x = \dfrac{1}{100}$ is $\dfrac{2\sqrt{6}}{5}$.

c Hence, determine $\sqrt{6}$ to 5 decimal places.

7 a Find the first three terms of the binomial expansion of $\dfrac{1}{\sqrt{1-2x}}$ where $|x| < \dfrac{1}{2}$.

b Hence or otherwise, obtain the expansion of $\dfrac{(2+3x)^3}{\sqrt{1-2x}}$, $|x| < \dfrac{1}{2}$.

Chapter summary

- Sequences may be **finite** or **infinite**.
- If the difference between two consecutive numbers in a sequence is constant then it is an **arithmetic sequence** or an **arithmetic progression**. The constant difference is called the **common difference** and is denoted by d.
- An arithmetic sequence with first term u_1 and common difference d has **general term** $u_n = u_1 + (n-1)d$.
- The sum of a finite arithmetic series is given by $S_n = \dfrac{n}{2}[2u_1+(n-1)d] = \dfrac{n}{2}[u_1+u_n]$ where n is the number of terms in the series, u_1 is the first term, d is the common difference and u_n is the last term.

Continued on next page

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- If the ratio of two consecutive terms in a sequence is constant then it is a **geometric sequence** or a **geometric progression**. We call the constant ratio the **common ratio** and denote it by r.
- A geometric sequence with first term u_1 and common ratio r has **general term** $u_n = u_1 r^{n-1}$, $r \neq 1,\ 0, -1,\ u_1 \neq 0$.
- The sum of a finite geometric series is given by

  $S_n = \frac{u_1(1-r^n)}{1-r}, \quad r \neq 1$

  where n is the number of terms, u_1 is the first term and r is the common ratio.
- The sum of n terms of a geometric series is $S_n = \frac{u_1(1-r^n)}{1-r}, \quad r \neq 1.$

  When $-1 < r < 1$, r^n approaches zero for very large values of n. The series therefore converges to a finite sum given by $S = \frac{u_1}{1-r}$.
- A **proof** in mathematics often consists of a logical set of steps that validates the truth of a general statement beyond any doubt.
- A **direct proof** is a way of showing the truth of a given statement by constructing a series of reasoned connected established facts. In a direct proof the following steps are used:
  - Identify the given statement.
  - Use axioms, theorems, etc, to make deductions that prove the conclusion of your statement to be true.
- When setting out a **proof by contradiction** you follow the following steps:
  - Identify what is being implied by the statement.
  - Assume that the implication is false.
  - Use axioms, theorems, etc … to arrive at a contradiction.
  - This proves that the original statement must be true.
- A **counterexample**, or **counterclaim**, is an acceptable "proof" of the fact that a given statement is false.
- $n! = n \times (n-1)! = n \times (n-1) \times (n-2) \times \ \ldots \ \times 3 \times 2 \times 1$
- The number of ways of arranging n distinct objects in a row is $n!$
- The number of permutations of r objects out of n distinct objects is given by ${}^nP_r = \frac{n!}{(n-r)!}$.
- The number of ways of choosing (when order is not important) r objects from n distinct objects is ${}^nC_r = \frac{n!}{r!(n-r)!}$

- The binomial expansion for $(1-x)^{-n}$ for $n \in \mathbb{Z}^+$ and $-1 < x < 1$ is given by the infinite series:

  $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots$
- The binomial expansion for $(1+x)^\alpha$ for $\alpha = \frac{p}{q} \in \mathbb{Q}$ and $-1 < x < 1$ is given by the infinite series:

  $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots + \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-r+1)}{r!}x^r + \ldots$

Developing inquiry skills

<To come>

Chapter review

Click here for mixed review exercise

1 Show that there are two geometric sequences such that the second term is 9 and the sum of the first three terms is 91. Write the fourth term of each sequence.

2 Find the sum of the series $1 + 2 + 3 + 4 + 5 + 7 + 8 + 9 + 11 + 13 + 15 + 16 + 17 + \ldots + 64$.

3 Three numbers a, b and c form an arithmetic sequence. The numbers b, c and a form a geometric sequence. Find the three numbers given that they add up to 36.

4 **a** Prove the identity $\frac{1}{1+x} - \frac{1}{3\left(1+\frac{2}{3}x\right)} \equiv \frac{x+2}{2x^2+5x+3}$.

b Hence, use the binomial expansion to find the first four terms of the expansion of $\frac{x+2}{2x^2+5x+3}$.

5 Prove the following identities:

a ${}^{n+1}C_2 \equiv {}^nC_2 + n$

b ${}^nC_2 \times {}^{n-2}C_{k-2} \equiv {}^nC_k \times {}^kC_2$

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6 Show that ${}^nC_0 + 3 \times {}^nC_1 + 3^2 \times {}^nC_2 + \ldots + 3^r \times {}^nC_r + \ldots + 3^n \times {}^nC_n = 2^{2n}$

7 Prove by contradiction that no two integers a and b can be found such that $14a + 7b = 1$.

8 Prove by contradiction that if $x = 3$ then $5x - 7 \neq 13$.

9 Give a counterexample to prove that each of the following statements is false:

a If $a^2 - b^2 < 0$ then $a - b > 0$.

b $3^n + 2$ is prime for all $n \in \mathbb{Z}^+$.

c $\sqrt{2n-1}$ is irrational for all $n \in \mathbb{Z}^+$.

d $2^n - 1$ is prime for all $n \in \mathbb{Z}^+$.

10 Prove by mathematical induction that $(1 \times 1!) \times (2^2 \times 2!) \times (3^3 \times 3!) \times (4^4 \times 4!) \times \ldots \times (n^n \times n!) = (n!)^{n+1}$

11 Use mathematical induction to prove that $n^3 + 2n$ is a multiple of 3.

12 Use mathematical induction to prove the following statements:

a $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ **b** $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

c $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$

Hence prove that

$$\sum_{r=1}^{n} r(r+1)(r+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

13 **a** In how many ways can the letters of the word *harmonics* be arranged?

b Determine how many numbers bigger than 30 000, less than 9 999 999 and divisible by 5 can be formed using the digits 0, 1, 2, 3, 5, 8 and 9.

c In how many ways can a committee of five people be selected from seven men and four women, so that there is at least one male and one female and there are more women than men on the committee?

14 Let $a = x + y$ and $b = x - y$.

a Write $a^2 - b^2$ in terms of x and y and hence show that $a^2 - b^2 = (a - b)(a + b)$.

b Use the binomial theorem to write a^3 and b^3 in terms of x and y.

c Use your results to part **b** to show that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

d Use the binomial theorem to write a^4 and b^4 in terms of x and y and use your result to factorize $a^4 - b^4$.

e Use your results to make a conjecture for the factors of $a^n - b^n$.

f Prove your conjecture using mathematical induction.

15 Given that the coefficients of x^{r-1}, x^r and x^{r+1} in the expansion of $(1 + x)^n$ are in arithmetic sequence, show that $n^2 + 4r^2 - 2 - n(4r + 1) = 0$.

Hence find three consecutive coefficients of the expansion of $(1 + x)^{14}$ which form an arithmetic sequence.

16 Given that

$$\frac{2 + x - 7x^2}{(1 - 2x)(1 - x^2)} \equiv \frac{A}{(1 - 2x)} + \frac{B}{(1 + x)} + \frac{C}{(1 - x)},$$

determine the values of A, B and C.

Hence use the binomial theorem to find the expansion of $\frac{2 + x - 7x^2}{(1 - 2x)(1 - x^2)}$ in ascending powers of x up to and including the term in x^3.

Exam-style questions

17 Find the coefficient of the term in x^5 in the binomial expansion of $(3 + x)(4 - 2x)^8$. (4 marks)

18 The coefficient of x^2 in the binomial expansion of $(1 + 3x)^n$ where $n \in \mathbb{Q}$ is 495.

Determine the possible values of n. (6 marks)

19 Find the value of $\sum_{n=0}^{n=15} (1.6^n - 12n + 1)$, giving your answer correct to 1 decimal place. (6 marks)

20 Prove the binomial coefficient identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. (6 marks)

21 Find the sum of all integers between 500 and 1400 (inclusive) that are not divisible by 7. (7 marks)

22 Prove by contradiction that for all $n \in \mathbb{Z}^+$, if $n^3 + 3$ is odd, then n is even. (7 marks)

23 Prove, by mathematical induction, that $5^{2n-1} + 1$ is divisible by 6 for all $n \in \mathbb{N}$. (8 marks)

24 **a** Find the first four terms, in ascending powers of x, of the binomial expansion of $\sqrt[3]{(1 - x)}$ $(|x| < 1)$. (4 marks)

b Use your answer to part a) to find an approximation for $\sqrt[3]{63}$ to six decimal places. You must show all your working. (5 marks)

25 Seven women and two men are chosen to sit in a row and have their photograph taken.

a How many different ways can they be arranged? (1 mark)

b How many ways can they be arranged if the men must sit together? (2 marks)

c How many ways can they be arranged if the men must sit apart? (2 marks)

d How many ways can they be arranged if there must be at least two women separating the men? (3 marks)

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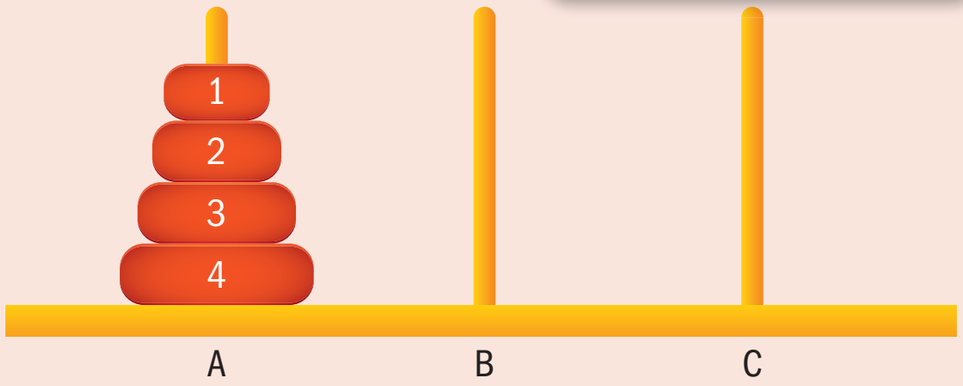
The Towers of Hanoi

Approaches to learning: Thinking skills, Communicating, Research

Exploration criteria: Mathematical communication (B) Personal engagement (C), Use of mathematics (E)

IB topic: Sequences

The problem



The aim of the **Towers of Hanoi problem** is to move all the disks from peg A to peg C following these rules:

1. Move only one disk at a time.
2. A larger disk may not be placed on top of a smaller disk.
3. All disks, except the one being moved, must be on a peg.

For 64 disks, what is the **minimum** number of moves needed to complete the problem?

Explore the problem

Use an online simulation to explore the Towers of Hanoi problem for 3 and 4 disks.

What is the minimum number of moves needed in each case?

Solving the problem for 64 disks would be very time consuming, so you need to look for a rule for n disks that you can then apply to the problem with 64 disks.

Try and test a rule

Assume the minimum number of moves follows an arithmetic sequence.

Use the minimum number of moves for 3 and 4 disks to predict the minimum number of moves for 5 disks.

Check your prediction using the simulator.

Does the minimum number of moves follow an arithmetic sequence?

Find more results

Use the simulator to write down the number of moves when $n = 1$ and $n = 2$.

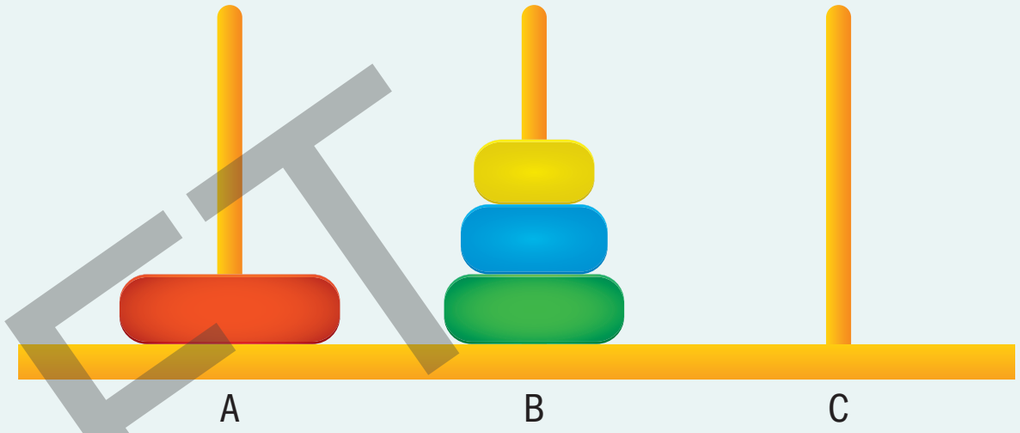
Organize your results so far in a table.

Look for a pattern. If necessary, extend your table to more values of n.

Try a formula

Return to the problem with 4 disks.

Consider this image of a partial solution to the problem. The large disk on peg A has not yet been moved.



Consider your previous answers.

What is the minimum possible number of moves made so far?

How many moves would it then take to move the largest disk from peg A to peg C?

When the large disk is on peg C, how many moves would it then take to move the 3 smaller disks from peg B to peg C?

How many total moves are therefore needed to complete this puzzle?

Use your answers to these questions to write a formula for the minimum number of moves needed to complete this puzzle with n disks.

This is an example of a **recursive formula**. What does that mean?

How can you check if your recursive formula works?

What is the problem with a recursive formula?

Try another formula

You can also try to solve the problem by finding an **explicit formula** that does not depend on you already knowing the previous minimum number.

You already know that the relationship is not arithmetic.

How can you tell that the relationship is not geometric?

Look for a pattern for the minimum number of moves in the table you constructed previously.

Hence write down a formula for the minimum number of moves in terms of n.

How does an explicit formula differ from a recursive formula?

Use your explicit formula to solve the problem with 64 disks.

Extension

- What would a solution look like for 4 pegs? Does the problem become harder or easier?
- Research the 'Bicolor' and 'Magnetic' versions of the Towers of Hanoi puzzle.
- Can you find an explicit formula for other recursive formulae? (e.g. Fibonacci)

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